Symplectic topology, day 2

Trisectors 2023
pre-talks

Austin Christian
Georgia Tech

Goals for the talks:

(1) Define symplectic & contact structures.

(2) Identify some examples and non-examples.

(3) Name important submanifolds of symplectic & contact manifolds.

(4) Determine the compatibility between these geometric structures and some topological constructions.

Today: A different take on (4).

Yesterday we asked symplectic geometry to play nicely with Morse theory and got a disappointing response.

Today we'll build up topological tools with which symplectic geometry *does* play nicely.

We'll focus exclusively on 4D.

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(Proof sketch) Let $\pi : X \to \Sigma_h$ be the surface bundle, with fiber Σ_g.

Let $\{U_i\}$ be a cover of Σ_h which trivializes π:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\sim", "\varphi_i"'] \arrow[dr, "p_i"'] & U_i \times \Sigma_g \arrow[d] \\ & \Sigma_g \end{tikzcd}$$

Use φ_i to define this.

$$
\begin{tikzcd}
\Sigma_g \arrow[r] & X \arrow[d, "\pi"] \\
& \Sigma_h
\end{tikzcd}
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$\Rightarrow \exists\, \eta \in \Omega^2(X)$ s.t. $\langle [\eta], [\Sigma_g] \rangle \neq 0.$

$\Rightarrow \exists$ symplectic $\sigma \in \Omega^2(\Sigma_g)$ s.t.

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- if $\omega \in \Omega^2(\Sigma_h)$ is symplectic, then so is $\tilde{\eta} + C\pi^*\omega$, for $C \gg 0$.

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- if $\omega \in \Omega^2(\Sigma_h)$ is symplectic, then so is $\tilde{\eta} + C\pi^*\omega$, for $C \gg 0$.

Vaguely: A surface bundle is locally a product $\Sigma_g \times \Sigma_h$ of symplectic mflds. We choose $C \gg 0$ so that the symplectic form from the base dominates the topology.

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Def A Lefschetz fibration is a smooth map $\pi : X \to \Sigma$ s.t.

- X is a cpct, connected, oriented 4-mfld;
- Σ is a cpct, connected, oriented surface;
- $\pi^{-1}(\partial\Sigma) = \partial X$;
- each C.P. of π is in $\text{int}(X)$, and is modeled on

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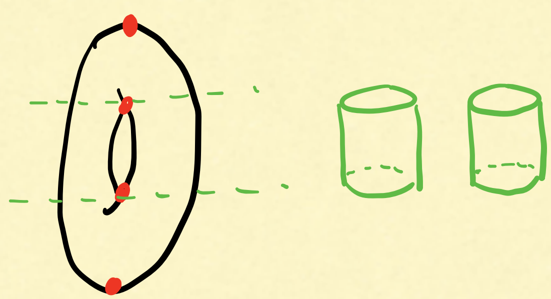

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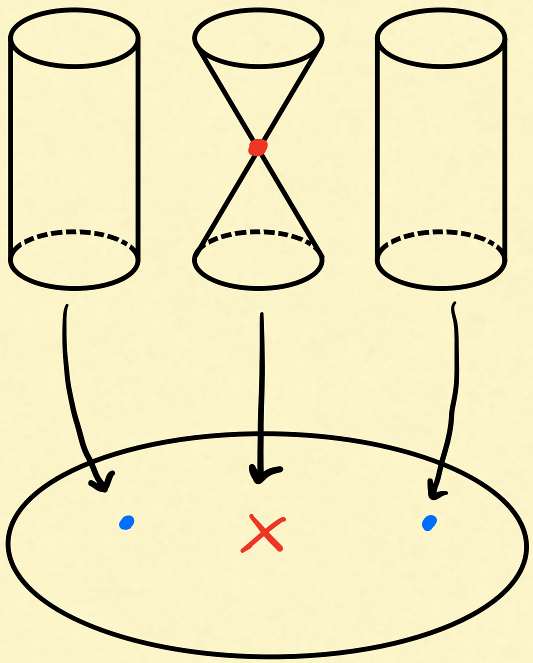
So the topology of X is bound up in the critical points, modeled on $0 = z_1^2 + z_2^2 = \underbrace{(z_1 + iz_2)}_{w_1}\underbrace{(z_1 - iz_2)}_{w_2}$.

Notice that $0=(z_1+iz_2)(z_1-iz_2)$ gives a pair of $\pitchfork$-intersecting planes, while $\zeta=\underbrace{(z_1+iz_2)}_{\in\mathbb{C}-\{0\}}(z_1-iz_2)$ gives a cylinder for $\zeta\neq 0$.



We will assume (possibly after a small perturbation) that each singular fiber contains exactly one C.P.

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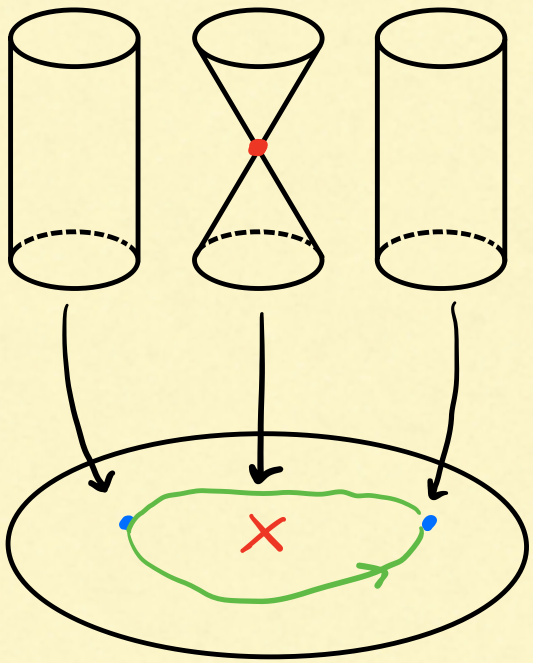

So understanding the local model means understanding the monodromy of

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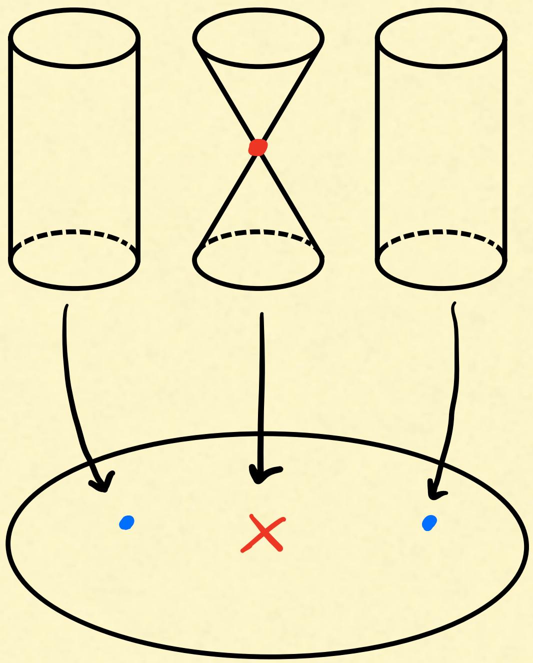

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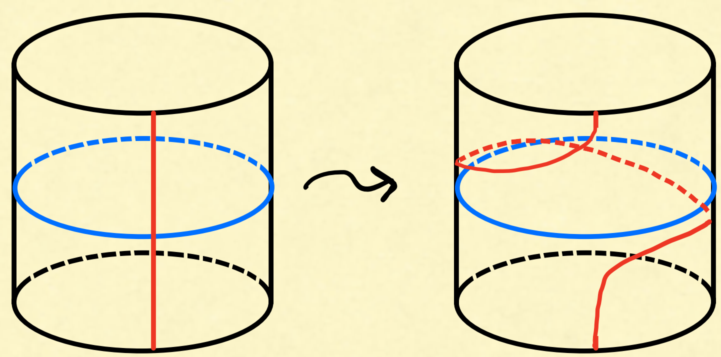
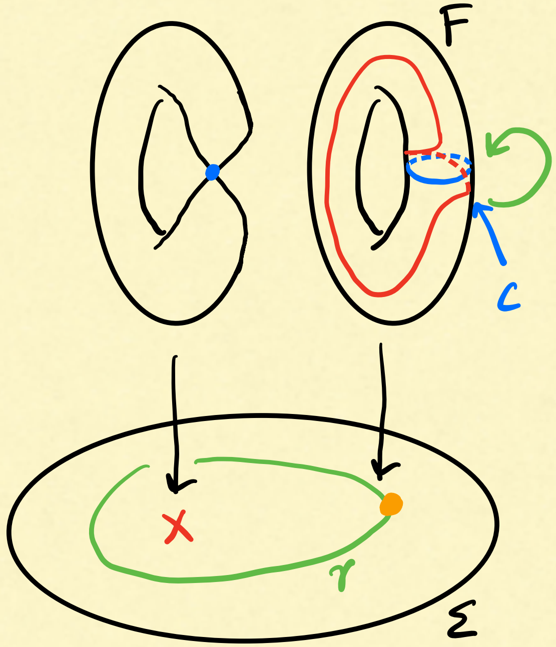
A (right-handed) Dehn twist looks like:



By investigating the local model, we find that, for any curve



$\gamma \subset \Sigma$ enclosing a single C.V. x, the monodromy around γ is a right-handed Dehn twist about some curve $c \subset F$.

We call c the vanishing cycle for x.

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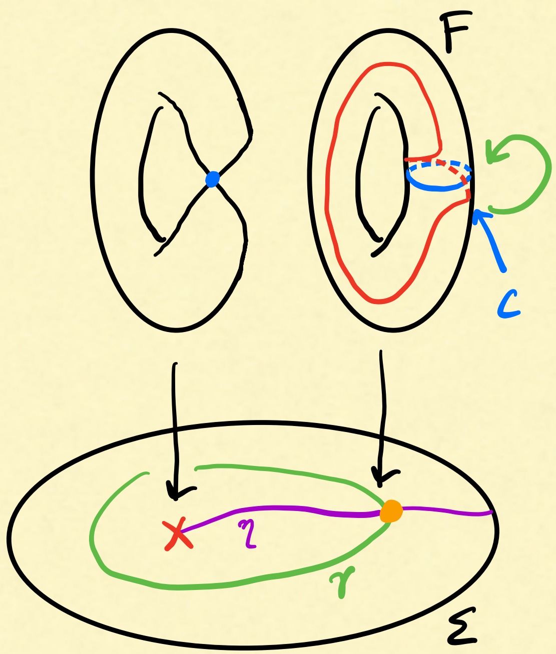

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Over a path η which abuts x, each regular fiber has a vanishing cycle; joining these with x gives a disc in X called the **thimble** over η.

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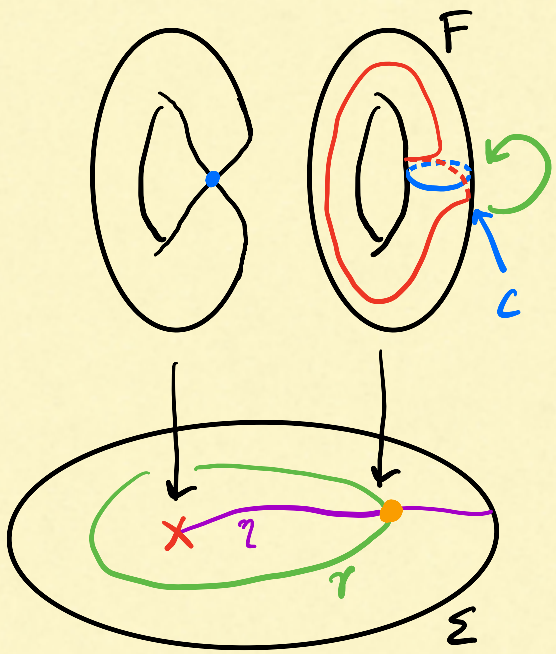

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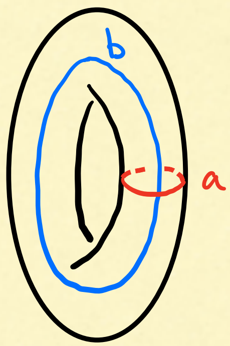
Over a path η which abuts $\times$, each regular fiber has a vanishing cycle; joining these with $\times$ gives a disc in X called the thimble over η.

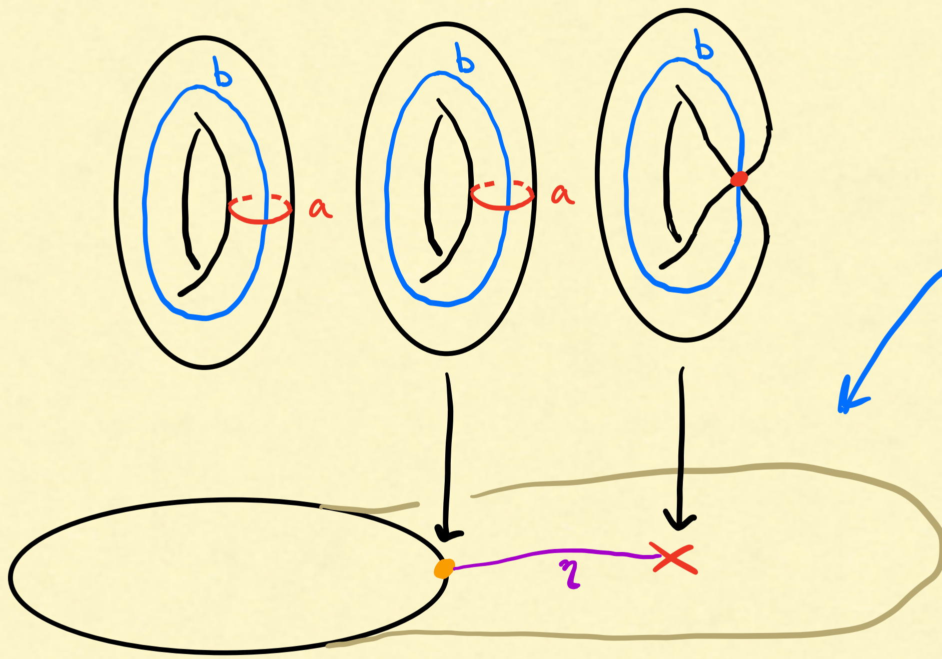
If η also abuts the boundary of our local model, then the thimble serves as the core of a 2-handle with attaching sphere $c \subset F$.

Ex. A Lefschetz fibration $\pi: X \to S^2$ with 12 genus 1 fibers.



Let $\tau_a, \tau_b: T^2 \to T^2$ denote right-handed Dehn twists.

Consider $X_0 = D^2 \times T^2$, with $\pi_0: X_0 \to D^2$.



2-handle attached along a. Core disc is thimble over η.

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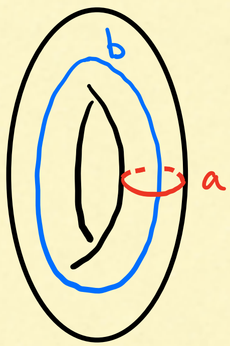

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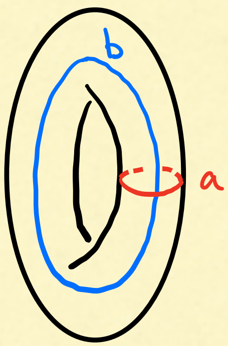

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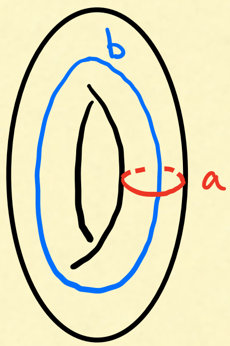

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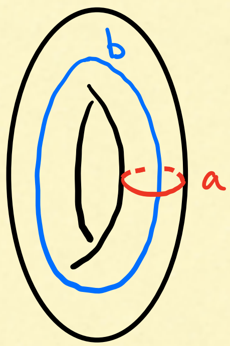

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__Fact__: $(\tau_a \tau_b)^6 \simeq id : T^2 \to T^2$, so we can cap off with $D^2 \times T^2$

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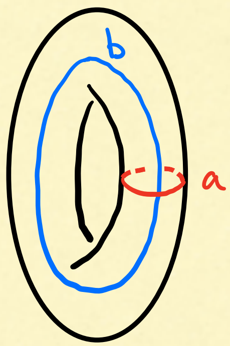

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Fact: $(\tau_a \tau_b)^6 \simeq id: T^2 \to T^2$, so we can cap off with $D^2 \times T^2$ to get $\pi: X \longrightarrow S^2$

$\cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$.

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Morally: As in Thurston's proof, construct $\eta \in \Omega^2(X)$ with $\langle [\eta], [\text{fiber}] \rangle \neq 0$ and set

$$\omega = \eta + C\pi^*\omega, \quad C \gg 0.$$

Over regular values of π, we again have $D^2 \times F$.

Over critical values we have a local model.

$\Rightarrow$ standard symplectic structure in which the singular fiber is symplectic.

Use a partition of unity to construct η globally.

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Ex. Let's try to build a L.F. $\pi : \mathbb{CP}^2 \to \mathbb{CP}^1 = S^2$.

Specifically, we want fibers of the form

$$C_t = \left\{ z \in \mathbb{CP}^2 \,\middle|\, (t_0 p_0 + t_1 p_1)(z) = 0,\ t = [t_0 : t_1] \right\},$$

for each $t \in \mathbb{CP}^1$, where p_0 & p_1 are generic homogeneous cubic polynomials.

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Alas, $\mathcal{C}_t \cap \mathcal{C}_{t'} = \{ p_0(z) = p_1(z) = 0 \}$
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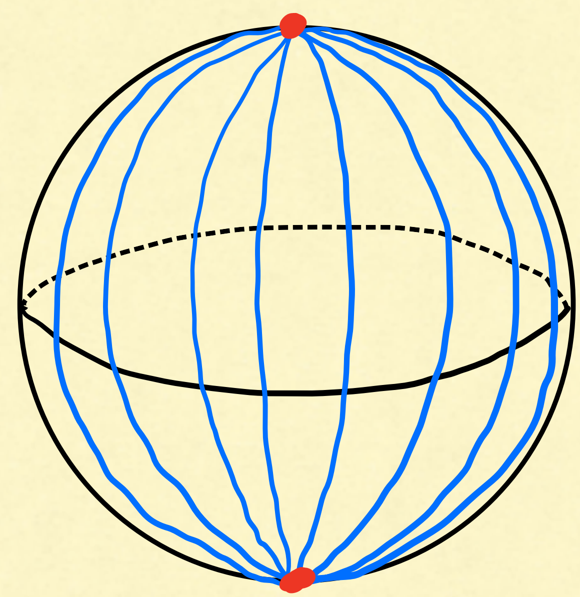

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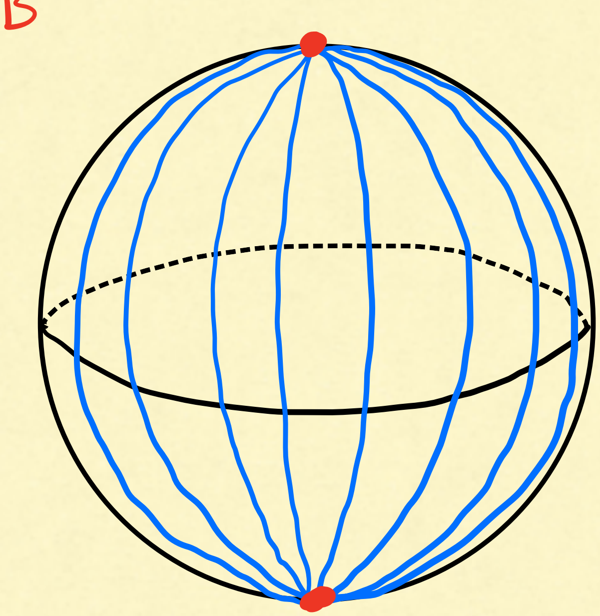

However, $\pi : \mathbb{C}P^2 \setminus B \to \mathbb{C}P^1$

behaves like a L.F. near its 12 critical points, and obeys a simple local model near B.

Def. A **Lefschetz pencil** is a smooth map

$$\pi : X \setminus B \to \mathbb{CP}^1 = S^2,$$

where
- X is a closed, connected, oriented 4-mfld;
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$$(z_1, z_2) \mapsto z_1^2 + z_2^2 ;$$

- near a point $b \in B$, π is modeled on

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Rmk. We motivated $z_1^2 + z_2^2$ via Morse theory. The motivation for z_1/z_2 is projective surfaces: pencils correspond to lines of hyperplane sections.

Def. A **Lefschetz pencil** is a smooth map

$$\pi : X \setminus B \to \mathbb{C}P^1 = S^2,$$

where
- X is a closed, connected, oriented 4-mfld;
- $B \subset X$ is a finite set of points;
- C.P.s of π are locally modeled on
$$(z_1, z_2) \mapsto z_1^2 + z_2^2;$$
- near a point $b \in B$, π is modeled on
$$(z_1, z_2) \mapsto z_1 / z_2.$$

Rmk. From a Lefschetz pencil Gay constructs a trisection s.t. each sector is a regular neighborhood of a regular fiber.

§4 ($\exists$ Lefschetz pencil) $\longleftrightarrow$ ($\exists$ symplectic form)

Good: If X^4 admits a Lefschetz pencil, then X admits a symplectic structure, canonical up to isotopy. (Gompf)

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Idea: Blow up X at each point of $B = \{b_1, \dots, b_n\}$ to obtain $X \# n\overline{\mathbb{CP}}^2$, which admits a Lefschetz fibration to $\mathbb{CP}^1$.

Now run previous argument, choosing $C >> 0$ large enough to make the exceptional spheres symplectic, blow down.

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Uses "approximately holomorphic sections" — the same technology used by Auroux ¡ Auroux-Katzarkov to study branched coverings $f: X \to \mathbb{CP}^2$, with X symplectic.

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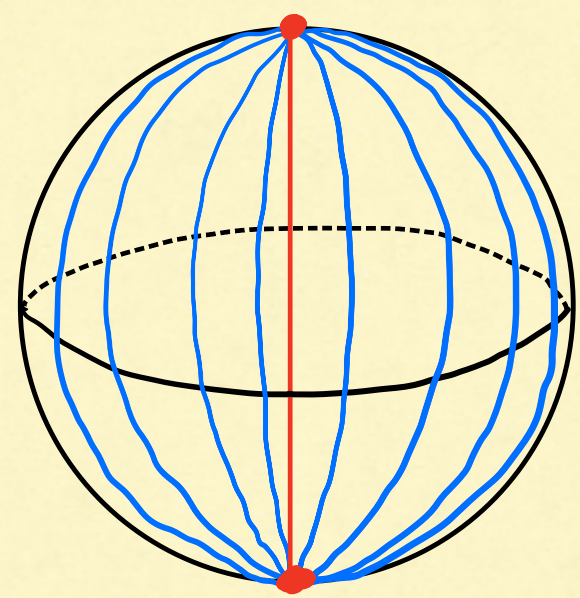
Cor.

$$\begin{array}{ccc} \{\text{symplectic 4-mflds}\} & & \\ \| & \subsetneq & \{\text{smooth 4-mflds}\} \\ \{\text{4-mflds admitting L.P.s}\} & \subsetneq & \end{array}$$

§5 The 3D contact story

The same story can be told for contact structures in 3D, with an additional topological miracle.



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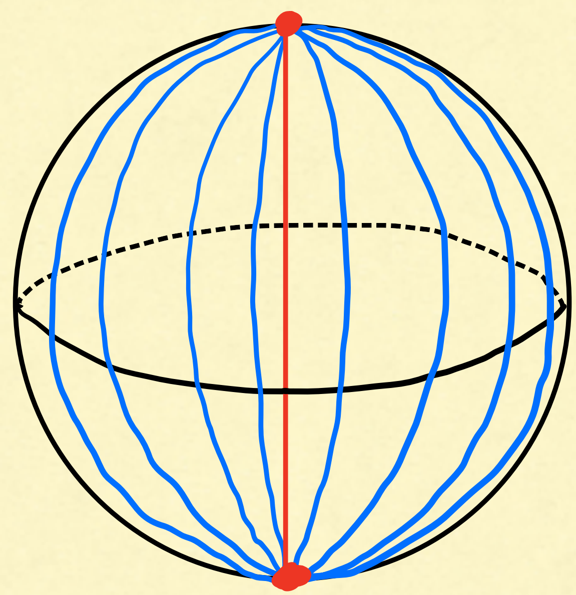
<u>Def</u>. An open book decomposition of Y^3 is a smooth fiber bundle

$$\pi : Y \setminus B \to S^1,$$

where $B \subset Y$ is an oriented link, and $\partial\left(\overline{\pi^{-1}(t)}\right) = B$, for each $t \in S^1$.



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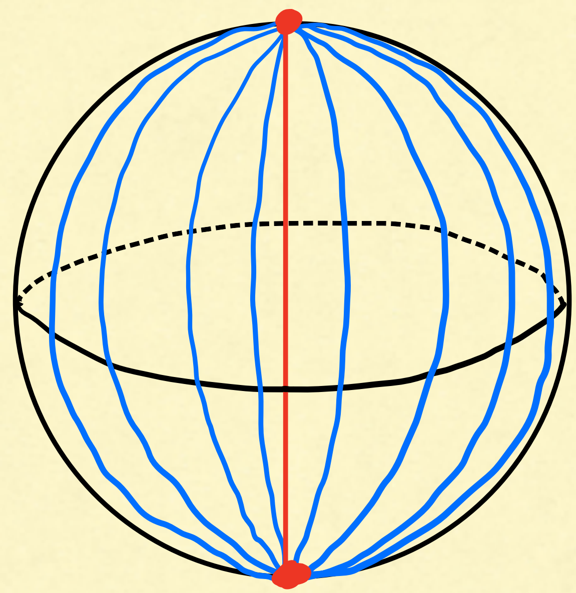

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__Thm__ (Alexander) Every closed, oriented 3-mfld admits an open book decomposition.

Thm (Thurston-Winkelnkemper) A contact structure may be constructed from any OBD of a 3-mfld.

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Same spirit as before: Base of fibration admits a contact form which can be used to dominate the topology.

Here it's important that the fibers are not only symplectic, but exact symplectic: $\omega = d\lambda$.

$$\text{1-form upstairs constructed from symplectic potentials} \longrightarrow \eta + K\pi^{*}\alpha$$

α: std contact form on S^1

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Same spirit as before: Base of fibration admits a contact form which can be used to dominate the topology.

Cor. (Lutz, Martinet) Every closed, oriented Y^3 admits ξ.

Thm (Giroux) Every contact 3-mfld admits a supporting OBD, unique in some sense.

up to stabilization of OBDs

In higher dimensions, every Y^{2n+1} admits an OBD, but T.W. argument no longer works — $\exists$ Y^{2n+1} w/ no ξ.
But! Every (Y^{2n+1}, ξ) admits supporting OBD.