Symplectic topology, day 1

Trisectors 2023
pre-talks

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Goals for the talks:

(1) Define symplectic & contact structures.

(2) Identify some examples and non-examples.

(3) Name important submanifolds of symplectic & contact manifolds.

(4) Determine the compatibility between these geometric structures and some topological constructions.

<u>Today</u>: (1)–(3) and a failure in (4)

§1 Symplectic & Contact structures

A classical mechanics fairytale:

Lagrangian mechanics		Hamiltonian mechanics
Potential function $M \longrightarrow \mathbb{R}$	$\xrightarrow[\text{transform}]{\text{Legendre}}$	Hamiltonian $T^*M \to \mathbb{R}$
Rmn gradient		Symplectic gradient

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In T^*M we have position and momentum coordinates.

We may give T^*M local coordinates $(\underset{\text{position}}{x}, \underset{\text{momentum}}{y})$.

We may give T^*M local coordinates (x, y).

position

momentum

Controversy!

We may give T^*M local coordinates (x, y).

In these coordinates, the standard symplectic form is $\omega_0 = dx \wedge dy \in \Omega^2(T^*M)$, and

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In general, a symplectic form on X^{2n} is a 2-form $\omega \in \Omega^2(X)$ which is

(1) closed: $d\omega = 0$;

(2) non-degenerate: $\overbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}^{n \text{ times}} \neq 0$.

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<u>Ex.</u> The standard area form $\omega = dA \in \Omega^2(S^2)$ trivially satisfies $d\omega = 0$ & $\omega \neq 0$.

Note that S^2 is not a cotangent bundle.

Instead, (S^2, dA) is the "classical counterpart" for the quantum notion of spin.

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In odd dimensions we have:

A (coorientable) contact structure on Y^{2n+1} is a codimension 1 distribution $\xi \subset TY$ s.t.

$$\xi = \operatorname{Ker} \alpha$$

for some $\alpha \in \Omega^1(Y)$ satisfying $\alpha \wedge (d\alpha)^n > 0$.

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<u>Ex.</u> On $\mathbb{R}_z \times T^*M$, let $\xi = \operatorname{Ker}(dz - y\,dx)$.

This is a natural setting for studying differential relations on M.

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The big question: Which smooth manifolds admit these structures?

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② Note that S^{2n} does **NOT** admit a symplectic form, for $n \geq 2$. If $\omega \in \Omega^2(S^2)$ were symplectic, we'd have

$$\omega^n \neq 0 \Rightarrow [\omega^n] \neq 0 \Rightarrow [\omega] \neq 0.$$

But $H^2(S^{2n}) = 0$.

③ Consider the standard symplectic form on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$:

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This passes to a 2-form ω_{FS} on $\mathbb{C}P^n$ via

$\iota^*\omega$ is invt. under $z \mapsto \lambda z$

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We call ω_{FS} the **Fubini-Study form** on $\mathbb{C}P^n$, and one can check that it's symplectic (and in fact makes $\mathbb{C}P^n$ Kähler).

On $\mathbb{C}P^2$, ω_{FS} is unique up to symplecto/scaling.

(4) The Fubini-Study form restricts to any nonsingular complex variety in $\mathbb{C}P^n$ to give a symplectic form.

Of particular interest: $\Sigma_d \subset \mathbb{C}P^2$.

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<u>Symplectic isotopy problem</u>: Which symplectic surfaces in $\mathbb{CP}^2$ are symplectically isotopic to C_d?

<u>Thm</u> (Siebert-Tian) If $V \subset (\mathbb{CP}^2, \omega_{FS})$ is a symplectic submfld with $[V] = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2)$, for some $d \leq 17$, then V is symplectically isotopic to C_d.

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Of particular interest: $\mathcal{C}_d \subset \mathbb{C}P^2$.

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Hope: Further progress via bridge trisections of symplectic surfaces in $(\mathbb{C}P^2, \omega_{FS})$.

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Hard problem: Understand the inclusions

$$\left\{\begin{matrix}\text{Complex}\\ \text{Varieties}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Symplectic}\\ \text{manifolds}\end{matrix}\right\} \subsetneq \left\{\begin{matrix}\text{Smooth}\\ \text{manifolds}\end{matrix}\right\}.$$

⑦ All 3-mflds admit contact structures (Lutz, Martinet).

One approach: • every 3-mfld admits an open book decomposition (Alexander)

• in 3D, OBD ⇒ contact structure (Thurston–Winkelnkemper)

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8. On $S^{2n+1} \subset \mathbb{C}P^{n+1}$ we may define $\xi := TS^{2n+1} \cap iTS^{2n+1}$.

Contrast with symplectic case.

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Contrast with symplectic case.

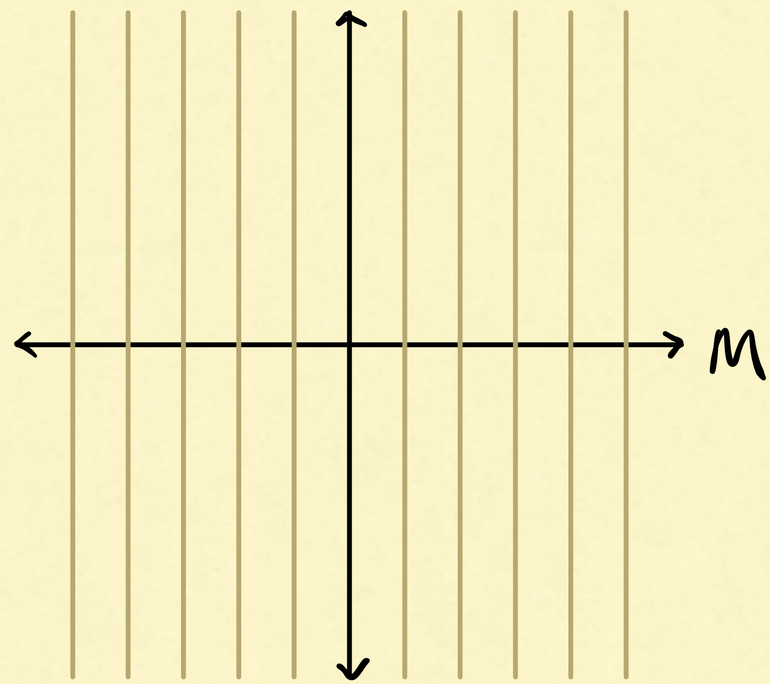
⑨ In dim > 3, the existence of a contact structure is a question of algebraic topology.

(Borman - Eliashberg - Murphy)

§3 Lagrangian & Legendrian submanifolds

In T^*M, neither the zero section M nor any fiber T_p^*M has symplectic area.

i.e., $\omega \equiv 0$ on these submanifolds



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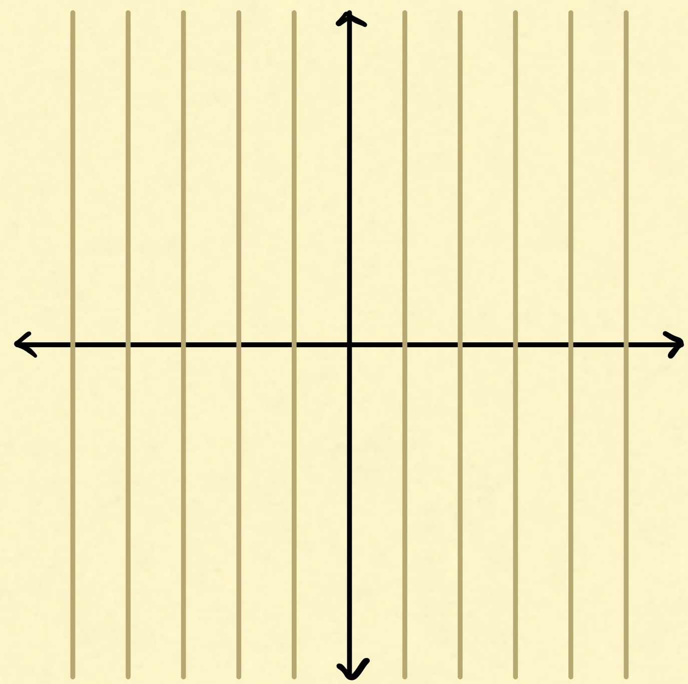

<u>Thm</u> (Weinstein) If (X^{2n}, ω) is symplectic and $L^n \subset X$ has $\omega|_L \equiv 0$, then $\exists$ a symplecto

$$\phi : (U, \omega_0) \longrightarrow (V, \omega),$$

where $V \subset X$ is a nbhd of L and $U \subset T^*L$ is a nbhd of the 0-section.

<u>Def.</u> We call $L^n \subset (X^{2n}, \omega)$ Lagrangian if $\omega|_L \equiv 0$.

<u>Idea</u>: If (X, ω) is locally phase space, then Lagrangians are locally position/momentum coords.

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Idea: If (X, ω) is locally phase space, then Lagrangians are locally position/momentum coords.

We have an analogous notion in contact manifolds:

Def. We call $\Lambda^n \subset (Y^{2n+1}, \xi)$ **Legendrian** if $T\Lambda \subset \xi$.

Thm. If $\Lambda^n \subset (Y^{2n+1}, \xi)$ is Legendrian, then $\exists$ a contacto

$$\phi : (U, \xi_0) \longrightarrow (V, \xi),$$

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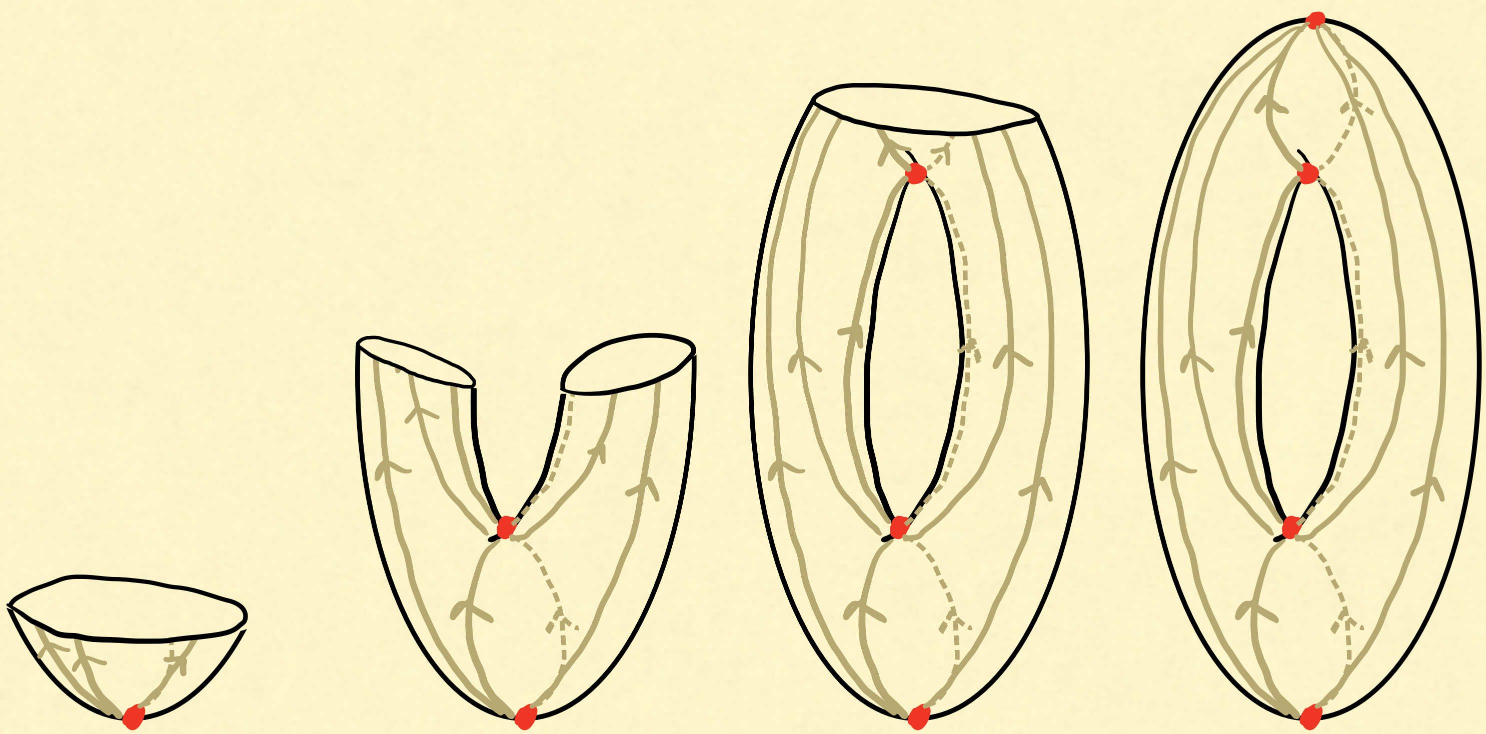
Today: (1)–(3) and a failure in (4)

We are here.

Recall the goal: Determine which smooth manifolds admit symplectic structures.

§4 Morse theory on symplectic & contact manifolds

There is a moral equivalence between Morse functions $f: X \to \mathbb{R}$ and handle decompositions of X.



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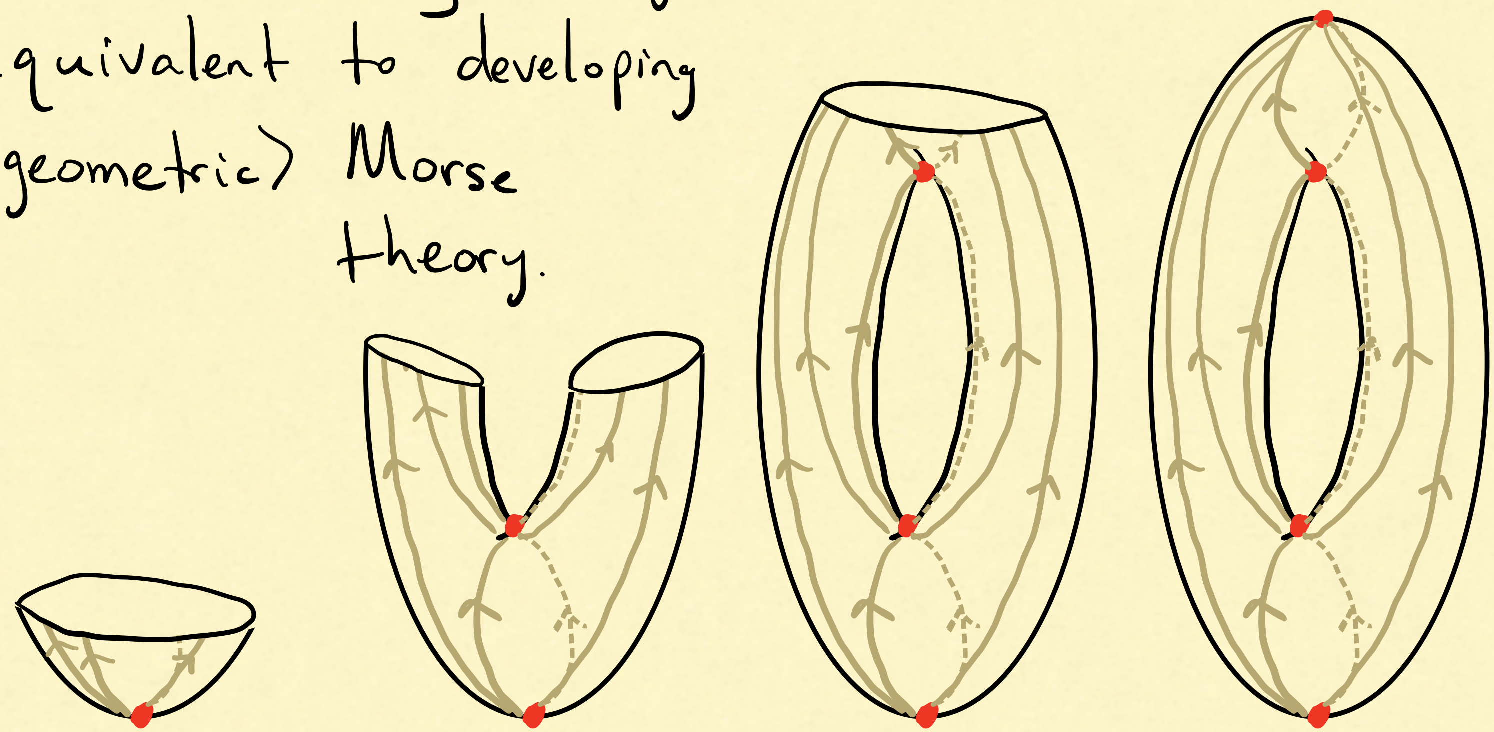
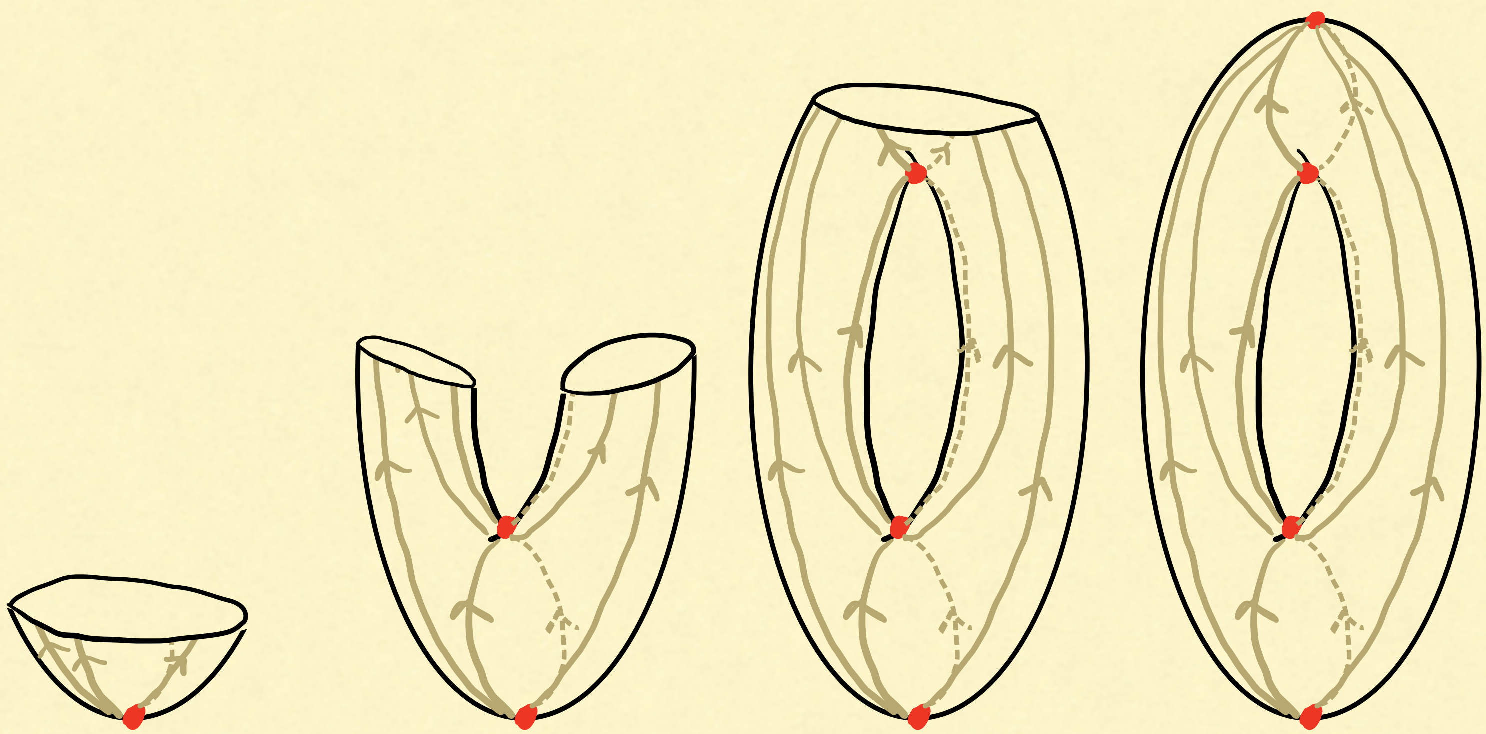
So constructing a (geometric) handle calculus is equivalent to developing (geometric) Morse theory.



A $\langle$geometric$\rangle$ Morse function on X should admit a gradient-like vector field whose flow $\phi_t : X \to X$ satisfies

$$\phi_t^* \langle \text{geometric structure} \rangle = \langle \text{geometric structure} \rangle.$$



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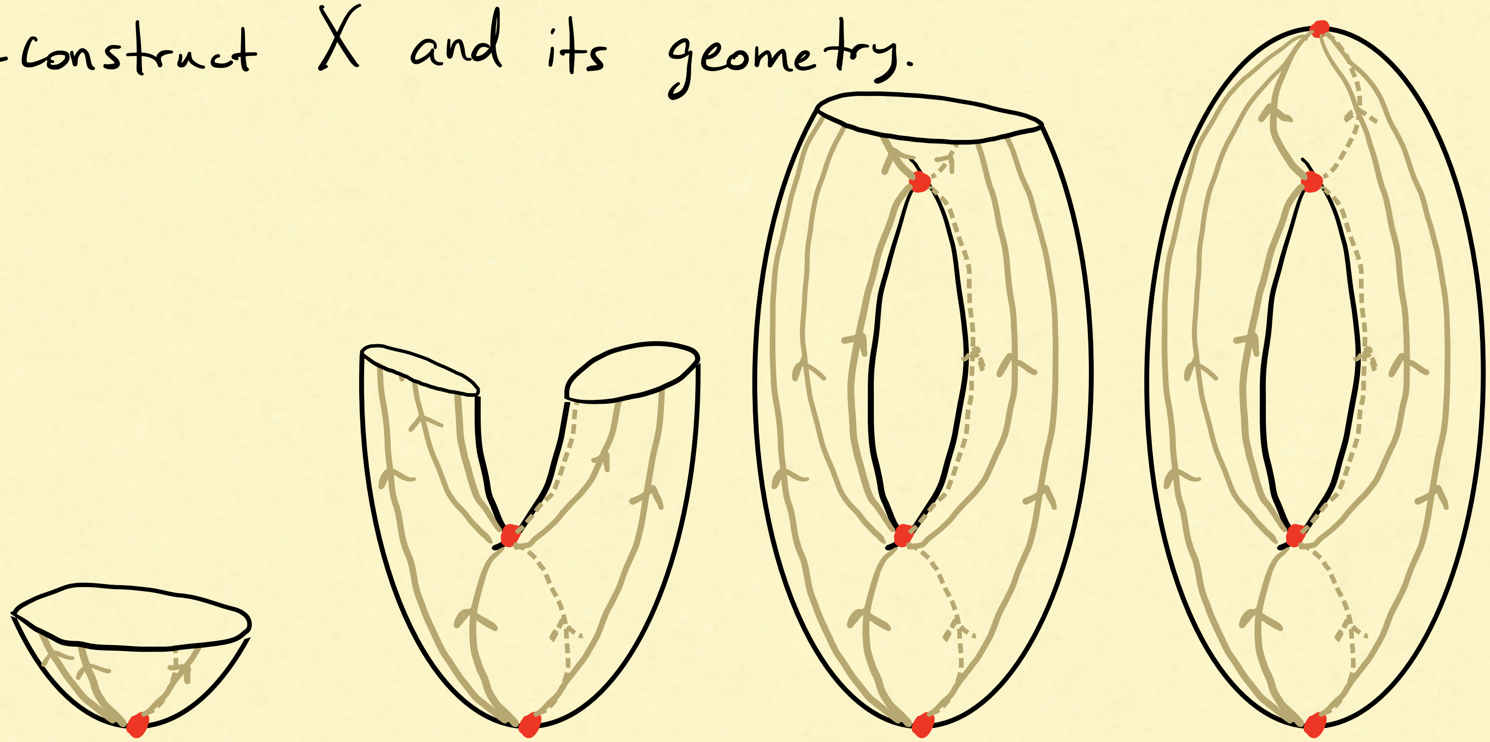
Thus $\{\text{critical points}\} \leftrightarrow \{\text{handles}\}$, and we can systematically reconstruct X and its geometry.



Def. Let (Y,ξ) be a contact manifold. A **contact Morse function** on Y is a Morse function $f: Y \to \mathbb{R}$ for which there is a vector field $V \in \mathfrak{X}(Y)$ satisfying:

(1) $df(V) > c\|V\|^2$, for some constant $c>0$ & some Rmn metric on Y;

(2) $\mathcal{L}_V \xi = 0$.

gradient-like vector field

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(2) $\underbrace{\mathcal{L}_V \xi = 0.}_{\text{"convex" contact structure}}$

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Symplectic manifolds are less fortunate.

Suppose (X^{2n}, ω) is symplectic, with $V \in \mathfrak{X}(X)$ satisfying

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Then $0 \neq \int_X \omega^n = \int_X (d\iota_V \omega)^n = \int_X d\eta = \int_{\partial X} \eta$, where

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<u>Upshot</u>. Closed symplectic manifolds do not admit "symplectic Morse functions" in the sense described here, and thus do not admit a systematic* symplectic handle calculus.

Indeed, the situation is worse than that:

<u>Fact</u>. If $f : X \to \mathbb{R}$ is Morse and admits $V \in \mathfrak{X}(X)$ which is gradient-like and satisfies

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then each C.P. of f has index at most n.

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Def. A Weinstein domain is a compact, symplectic manifold-with-boundary (X, ω) along with a choice of (1) vector field $V \in \mathfrak{X}(X)$ s.t. $\mathcal{L}_V \omega = \omega$;

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Thm. (Weinstein) Weinstein domains can be systematically constructed from Weinstein handles, where a Weinstein k-handle, $0 \le k \le n$, is modeled on $(\mathbb{R}^{2n}, \omega_0)$ and the Morse function $f_k : \mathbb{R}^{2n} \to \mathbb{R}$ has a single C.P. of index k.

A Weinstein n-handle is attached along a Legendrian sphere in ∂X.

In summary:

- Determining which smooth manifolds admit symplectic structures is hard, for contact structures it's algebraic topology.
- In the search for a (smooth) topological description of symplectic manifolds, handle calculus is insufficient.
- Tomorrow we'll discuss topological constructions more amenable to symplectic structures.