A SKETCH OF CONTACT HOMOLOGY

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1. Introduction

Recall that the last talk in this series developed a version of Legendrian contact homology for Legendrian knots in $\mathbb{R}^3$. Specifically, that talk presented the differential graded algebra defined by Chekanov in [6], following the exposition of [8]. A knot $K \subset \mathbb{R}^3$ was said to be Legendrian if it was everywhere tangent to the contact structure $\xi_0 = \ker \alpha_0$, where α_0 was the contact form $dz - ydx$. Given a Legendrian knot K, the Chekanov-Eliashberg DGA was then generated by the double points of $\pi(K)$, where $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$ was the xy-projection. The differential on this DGA was then defined by counting certain polygonal maps between the double points.

Though we did not emphasize this then, Legendrian contact homology is supposed to be a sort of Morse homology for an action functional. Consider the functional defined on paths in $\mathbb{R}^3$ by

$$\mathcal{A}(\psi) := \int_\psi \alpha_0.$$

The action-minimizing paths of this functional will have their tangent vectors in the kernel of $d\alpha_0$, and will thus be directed by the *Reeb vector field* of α_0, defined by the equations

$$\iota_{R_{\alpha_0}} \alpha_0 \equiv 1 \quad \text{and} \quad \iota_{R_{\alpha_0}} d\alpha_0 \equiv 0.$$

We quickly see that the Reeb vector field for $\alpha_0 = dz - ydx$ is ∂_z, so the action-minimizing paths are vertical lines. Thus the double points of $\pi(K)$ represent those pairs of points on K which are connected by an action-minimizing path. In this way the generators of our DGA correspond to critical points of the functional $\mathcal{A}$, and the differential then counts curves connecting one critical point to another (of course we're sweeping quite a lot under the rug here).

Our goal today will be to investigate these same ideas in the absence of a Legendrian subspace. That is, instead of a DGA generated by Reeb *chords* — the action-minimizing paths connecting distinct points on K — we will consider a DGA generated by Reeb *orbits*. Certainly $(\mathbb{R}^3, \xi_0)$ has no such orbits, but the Weinstein conjecture encourages us to expect Reeb orbits to exist for closed contact manifolds.

We plan to construct contact homology under ideal but very restrictive conditions. The primary reference for this talk is [3], and the interested reader would likely be better served by reading those notes. Given a contact manifold and some auxiliary data, we will construct a DGA whose homology will not depend on the associated data. As mentioned above, the algebra will be generated by a collection of Reeb orbits, and, as with any Floer-type theory, the differential will involve a count of J-holomorphic curves.

Throughout, we take (M^{2n-1}, ξ) to be a fixed compact contact manifold. Further, we assume that ξ is co-oriented, and denote by α a global contact form for ξ.

2. The graded algebra

2.1. **Critical points.** As with any Floer homology, contact homology mimics Morse homology, with a functional playing the role usually played by a Morse function. For contact homology, the functional of

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interest is the action functional $\mathcal{A}\colon C^\infty(S^1, M) \to \mathbb{R}$ defined by

$$\mathcal{A}(\gamma) := \int_\gamma \alpha,$$

where α is the contact form chosen above. If we hope to perform a Morse-type analysis on this functional, we should identify its critical points. Recall that a Reeb orbit is an orbit $\gamma\colon S^1 \to M$ for which $\iota_{\dot\gamma} d\alpha = 0$. A least action principle then tells us that $\gamma \in C^\infty(S^1, M)$ is a critical point of $\mathcal{A}$ if and only if γ is a closed Reeb orbit of $(M, \ker \alpha)$. We care especially about non-degenerate critical points of $\mathcal{A}$; we should determine what non-degeneracy means for closed Reeb orbits. Say $\gamma\colon \mathbb{R}/T\mathbb{Z} \to M$ is a closed Reeb orbit and let

$$\phi_t\colon M \to M$$

be the time t-Reeb flow. From this flow we obtain the *linearized return map*

$$\Psi_\gamma := d(\phi_T)_{\gamma(0)}|_{\xi_{\gamma(0)}}\colon \xi_{\gamma(0)} \to \xi_{\gamma(0)}.$$

The Reeb flow preserves the contact structure ξ, as well as its symplectic form $d\alpha$, so the return map is a symplectic linear map. One can show that γ is non-degenerate as a critical point of $\mathcal{A}$ if and only if the linearized return map Ψ_γ does not have 1 as an eigenvalue, and we will call orbits with this property *nondegenerate* Reeb orbits.

In order to mimic Morse theory, we want all critical points of our action functional to be non-degenerate. Thankfully, we have the following:

Lemma 1. *For any contact structure ξ on M, there exists a contact form α for ξ such that all closed orbits of R_α are non-degenerate.*

This is great in that it allows us to do Morse theory, but it comes at a cost. If we have a particularly nice contact form α (say, with lots of symmetry), we might be able to reasonably compute its Reeb flows. But by perturbing the original form we break this symmetry, perhaps making the Reeb flows unreasonable to compute. In [2], Bourgeois develops some alternative techniques for computing contact homology (via a different chain complex) that relaxes the non-degeneracy requirements to Morse-Bott-type requirements. We will not pursue these ideas here.

We will say that a Reeb orbit $\gamma\colon \mathbb{R}/T\mathbb{Z} \to M$ is *simple* if γ has minimal period among all Reeb orbits with image $\gamma(\mathbb{R}/T\mathbb{Z})$. If $\gamma\colon \mathbb{R}/T\mathbb{Z} \to M$ is simple, we will call the Reeb orbit

$$\gamma^k\colon \mathbb{R}/kT\mathbb{Z} \to M\colon t \mapsto \gamma(t)$$

the *k-fold iterate* or *k-fold multiple cover* of γ. For each simple closed Reeb orbit γ we also fix a *marker* $x_\gamma = \gamma(t_0)$. These markers will not affect the construction of our algebra, but will matter for our differential.

2.2. **An index.** The graded algebra of interest to us will be generated by non-degenerate, closed Reeb orbits, but not all such orbits. Defining the differential on our algebra will require a more narrow set of generators, and the additional criteria are most easily expressed via a condition on the grading. That is, we are going to define a grading on the closed Reeb orbits, and then use this grading to determine which orbits are admissible as generators of our algebra.

Suppose $\gamma\colon \mathbb{R}/T\mathbb{Z} \to M$ is a non-degenerate, closed Reeb orbit, and fix a trivialization of ξ along γ. With respect to this trivialization, the linearized Reeb flow $d\phi_t\colon \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ induces a path

$$\Psi_\gamma\colon [0, T] \to \mathrm{Sp}(2n - 2).$$

Notice that we previously used Ψ_γ to denote the return map that is now denoted by $\Psi_\gamma(T)$. Certainly $\Psi_\gamma(0) = I$, and since γ is non-degenerate, $\det(\Psi_\gamma(T) - I) \neq 0$. We consider the t-values for which degeneration occurs, which we call *crossings*:

$$\mathrm{Cross}_\gamma := \{t \in [0, T] | \det(\Psi_\gamma(t) - I) = 0\}.$$

At each crossing $t \in [0, T]$ we let $E_t := \ker(\Psi_\gamma(t) - I)$ and consider the quadratic form $\Gamma(\Psi_\gamma, t)\colon E_t \to \mathbb{R}$ defined by

$$v \mapsto d\alpha(v, \dot\Psi_\gamma(t)v).$$

We call a crossing $t \in \mathrm{Cross}_\gamma$ *regular* if $\Gamma(\Psi_\gamma, t)$ is non-degenerate. If all crossings of the path Ψ_γ are regular, we define the *Conley-Zehnder index* of Ψ_γ by

$$(1) \qquad \mu_{CZ}(\Psi_\gamma) = \frac{1}{2} \operatorname{sign} \Gamma(\Psi_\gamma, 0) + \sum_{t \in \mathrm{Cross}_\gamma \backslash \{0\}} \operatorname{sign} \Gamma(\Psi_\gamma, t).$$

We can extend our definition of the Conley-Zehnder index to all paths of symplectic matrices with the help of the following lemma of Robbin and Salamon [12]:

Lemma 2. *If $\Psi, \Psi' \colon [0, T] \to \mathrm{Sp}(2n-2)$ are homotopic with fixed endpoints and have regular crossings, then $\mu_{CZ}(\Psi) = \mu_{CZ}(\Psi')$.*

For an arbitrary path $\Psi \colon [0, T] \to \mathrm{Sp}(2n-2)$ with $\Psi(0) = I$ and $\det(\Psi(T) - I) \neq 0$, we may then choose Ψ' to be fixed-endpoint-homotopic to Ψ and have regular crossings, and define $\mu_{CZ}(\Psi) := \mu_{CZ}(\Psi')$.

As with the Maslov index, the Conley-Zehnder index could instead be defined as the degree of a loop in S^1, using a continuous extension of the determinant map $\mathrm{U}(n) \to S^1$ to all of $\mathrm{Sp}(2n)$. For an exposition of this approach, and to see that the two definitions are equivalent, consult [12] or [13].

Of course the Conley-Zehnder index of a closed Reeb orbit γ is not exactly well-defined, since it depends on the choice of a trivialization of ξ along γ. Its parity, however, is well-defined, and in fact can be deduced from the linearized return map via the following formula:

$$(2) \qquad (-1)^{\mu_{CZ}(\gamma)} = (-1)^{n-1} \operatorname{sign} \det(\Psi_\gamma(T) - I).$$

See section 1.2 of [7]. We call a Reeb orbit *even* (resp. *odd*) if its Conley-Zehnder index is even (resp. odd).

2.3. **An example.** Suppose $a, b > 0$ are positive real numbers, with a/b irrational, and consider

$$M = \left\{ (z_1, z_2) \,\middle|\, \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} = 1 \right\} \subset \mathbb{C}^2,$$

the boundary of an ellipsoid. Writing $z_j = x_j + iy_j$ for $j = 1, 2$, we give M the contact form

$$\alpha = \frac{1}{2} \sum_{j=1}^{2} (x_j dy_j - y_j dx_j).$$

The Reeb vector field for this form is easily seen to be

$$R = \frac{2\pi}{a} \partial_{\theta_1} + \frac{2\pi}{b} \partial_{\theta_2},$$

where $\partial_{\theta_j} = x_j \partial_{y_j} - y_j \partial_{x_j}$ for $j = 1, 2$. Because a/b is irrational, there are precisely two closed, simple Reeb orbits. Both are circles, and we consider now the closed Reeb orbit parametrized by

$$\gamma(t) = (\sqrt{a/\pi} \exp(2\pi i t/a), 0), \text{ for } t \in [0, a].$$

The 2-dimensional contact plane $\xi_{\gamma(t)}$ is spanned by ∂_{x_2} and ∂_{y_2}, for all $t \in [0, a]$, and this provides a trivialization of ξ along γ. The linearized Reeb flow $d\phi_t \colon \xi_{\gamma(0)} \to \xi_{\gamma(t)}$ is given by rotation through an angle of $2\pi t/b$, which has an eigenvalue of 1 precisely when t is an integer multiple of b. So

$$\mathrm{Cross}_\gamma = \{kb \,|\, k \in \mathbb{Z}, kb \in [0, a]\},$$

and thus $|\mathrm{Cross}_\gamma| = \lfloor \frac{a}{b} \rfloor + 1$. Notice that because a/b is irrational, a is not a crossing, so γ is a non-degenerate closed Reeb orbit, as are all of its iterates. The next step in computing the Conley-Zehnder index of γ requires an investigation of the quadratic forms $\Gamma(\Psi_\gamma, t)$, where $t \in \mathrm{Cross}_\gamma$. To this end, we note that for $t \in \mathrm{Cross}_\gamma$ we have $\Psi_\gamma(t) = I$ and

$$\dot{\Psi}_\gamma(t) = \frac{2\pi}{b} \begin{bmatrix} -\sin(2\pi t/b) & -\cos(2\pi t/b) \\ \cos(2\pi t/b) & -\sin(2\pi t/b) \end{bmatrix} = \frac{2\pi}{b} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \frac{2\pi}{b} J_0.$$

Now since $d\alpha = \sum_{j=1}^{2} dx_j \wedge dy_j$, we have $d\alpha = dx_2 \wedge dy_2$ on $\xi_{\gamma(t)}$, so

$$d\alpha(v, \dot{\Psi}_\gamma(t)v) = \frac{2\pi}{b} (dx_2 \wedge dy_2)(v, J_0 v) = \frac{2\pi}{b} \|v\|^2$$

for every $v \in \xi_{\gamma(t)} = \ker(\Psi_\gamma(t) - I)$. In particular this means that $\Gamma(\Psi_\gamma, t)$ has signature 2 for all crossings $t \in \mathrm{Cross}_\gamma$, and we conclude that the Conley-Zehnder index of γ *with respect to our chosen trivialization*[1] is

$$\mu_{CZ}(\Psi_\gamma) = 1 + \sum_{t \in \mathrm{Cross}_\gamma \setminus \{0\}} 2 = 2 \left\lfloor \frac{a}{b} \right\rfloor + 1.$$

This example indicates the rough idea one should have for the Conley-Zehnder index, at least in dimension three: given a non-degenerate Reeb orbit γ and a trivialization of ξ along this orbit, the Conley-Zehnder index is a measurement of the rotation of the linearized Reeb flow around γ with respect to the trivialization.

2.4. **A word about elliptic/hyperbolic orbits.** We momentarily restrict our attention to the $n = 2$ case, so that M is a 3-dimensional manifold. Given a non-degenerate closed Reeb orbit $\gamma \colon \mathbb{R}/T\mathbb{Z} \to M$, its linearized return map $\Psi_\gamma \colon \xi_{\gamma(0)} \to \xi_{\gamma(0)}$ is an automorphism of the two-dimensional symplectic vector space $(\xi_{\gamma(0)}, d\alpha)$. Because symplectic maps have determinant 1 and have the property that their eigenvalues appear in pairs λ, λ^{-1}, we may choose $\lambda \in \mathbb{C} \setminus \{0\}$ so that λ and λ^{-1} are the eigenvalues of Ψ_γ. Moreover, because γ is non-degenerate, $\lambda \neq 1$. We now have three cases: either $\lambda \notin \mathbb{R}$, $\lambda > 0$, or $\lambda < 0$. In the first case, Ψ_γ has distinct eigenvalues $e^{i\theta}, e^{-i\theta}$, so Ψ_γ is conjugate to a rotation in $\xi_{\gamma(0)}$ through an angle of $\pm\theta$. In this case we say that γ is an *elliptic* Reeb orbit. We may use equation (2) to determine the parity of the Conley-Zehnder index of γ. We have

$$(-1)^{\mu_{CZ}(\gamma)} = (-1)^1 \operatorname{sign}(2 - 2\cos\theta) = -1,$$

so $\mu_{CZ}(\gamma)$ is odd for all elliptic Reeb orbits γ. Notice that if all iterates of γ are to be non-degenerate, then $2\pi/\theta$ must be irrational.

Next we suppose that $\lambda \in \mathbb{R}$. If $\lambda = -1$, the fact that Ψ_γ is symplectic tells us that $\Psi_\gamma = -I$, in which case γ^2 is degenerate. Since we are operating under the assumption that no closed Reeb orbits are degenerate, this cannot happen. So we have $\lambda \neq -1$, meaning that Ψ_γ has distinct eigenvalues λ, λ^{-1}, and thus $\xi_{\gamma(0)}$ admits a basis $\{v_1, v_2\}$ with respect to which we may write

$$\Psi_\gamma(T) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}.$$

We lose no generality by assuming that $|\lambda| < 1$, in which case v_1 and v_2 span stable and unstable subspaces of $\xi_{\gamma(0)}$, respectively. We call all Reeb orbits γ for which Ψ_γ has only real eigenvalues *hyperbolic*; we prepend the adjective *positive* or *negative* in case $\lambda > 0$ or $\lambda < 0$. For the parity of the Conley-Zehnder index we have

$$(-1)^{\mu_{CZ}(\gamma)} = (-1)^1 \operatorname{sign}((\lambda - 1)(\lambda^{-1} - 1)) = \operatorname{sign}(\lambda),$$

since $|\lambda| < 1$. So we see that $\mu_{CZ}(\gamma)$ is even if γ is positive hyperbolic and odd if γ is negative hyperbolic.

Suppose γ is negative hyperbolic, and that $v \in \xi_{\gamma(0)}$ is an eigenvector of the linearized return map Ψ_γ with eigenvalue $-1 < \lambda < 0$. Because $|\lambda| < 1$, the iterates $\Psi_\gamma^k v$ tend to 0 as k tends to infinity, and perhaps we think of the line spanned by v as a stable subspace of $\xi_{\gamma(0)}$. Notice, however, that the orientations of $\Psi_\gamma^{2k} v$ and $\Psi_\gamma^{2k+1} v$ disagree, indicating that we might run into trouble if we attempt a Morse theory which uses both even and odd iterates of γ as generators. We will soon write down a condition that excludes Reeb orbits which create these kinds of problems.

2.5. **A grading.** For reasons that will remain opaque, we alter the Conley-Zehnder index to produce a grading on the closed Reeb orbits. We define

$$|\gamma| := \mu_{CZ}(\gamma) + n - 3.$$

As mentioned above, the definition of the Conley-Zehnder index requires that we choose a trivialization for ξ along γ, so our grading is not entirely well-defined. As with the Conley-Zehnder index, however, the parity is well-defined. This allows us to give a simple test for the admissibility of a closed Reeb orbit γ. Let γ_s be the simple closed Reeb orbit underlying γ. We say that γ is *good* if the parities of $|\gamma|$ and $|\gamma_s|$ agree, and

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[1]As it happens, computing the contact homology of M would require a different trivialization, and thus produce a different Conley-Zehnder index.

say that γ is *bad* otherwise. Notice that dimension three, the bad Reeb orbits are precisely the even covers of negative hyperbolic orbits — the orbits we suspected would cause problems.

2.6. **A Novikov ring.** Suppose $\gamma = \partial\Sigma_\gamma$ and trivialize ξ on Σ_γ to obtain $\mu_{CZ}(\gamma; \Sigma_\gamma)$, the Conley-Zehnder index of γ with respect to this spanning surface. If we have $[A] \in H_2(M, \mathbb{Z})$, we can instead trivialize ξ on the connected sum $\Sigma_\gamma \sharp A$ to obtain a different Conley-Zehnder index $\mu_{CZ}(\gamma; \Sigma_\gamma \sharp A)$, and these indices are related by the first Chern class of ξ:

$$(3) \qquad \mu_{CZ}(\gamma; \Sigma_\gamma \sharp A) = \mu_{CZ}(\gamma; \Sigma_\gamma) + 2\langle c_1(\xi), [A]\rangle.$$

A proof of the index formula (3) can be found in Appendix A of [11]. This formula captures some of the ambiguity of the grading we have chosen for the generators of our algebra, and it will inspire a grading on the coefficient ring of our algebra.

The coefficient ring of our algebra will be a group ring of $H_2(M, \mathbb{Z})$. The index formula (3) suggests the grading

$$(4) \qquad |A| = -2\langle c_1(\xi), [A]\rangle$$

on $H_2(M, \mathbb{Z})$, and we will take the group ring $\mathbb{Q}[H_2(M, \mathbb{Z})]$ as our coefficient ring. The elements of $\mathbb{Q}[H_2(M, \mathbb{Z})]$ will be written as

$$\sum_{i=1}^{k} q_i e^{A_i},$$

with $q_i \in \mathbb{Q}$ and $A_i \in H_2(M, \mathbb{Z})$. For further discussion of why this should be the coefficient ring of our graded algebra, see section 9.2 of [10].

Finally, we let A be the graded, unital, supercommutative algebra freely generated by all good closed Reeb orbits over the group ring $\mathbb{Q}[H_2(M, \mathbb{Z})]$. Supercommutativity means that $\gamma_1\gamma_2 = (-1)^{|\gamma_1||\gamma_2|}\gamma_2\gamma_1$ for all generators γ_1, γ_2.

3. The differential

As with any Floer-type homology theory, the differential we define on the graded algebra A will involve counting J-holomorphic curves of special interest to us. Naturally, performing such a count will require the relevant moduli spaces to be compact and fairly regular, and the differential will consider the sizes of the one-dimensional moduli spaces, modulo an $\mathbb{R}$-action.

3.1. **The J-holomorphic curves.** The J-holomorphic curves of interest will land in the *symplectization*

$$(\mathbb{R} \times M, d(e^t\alpha))$$

of the co-oriented contact manifold $(M^{2n-1}, \ker\alpha)$. The almost complex structure we consider on the symplectization will arise from a complex structure on the contact distribution ξ. If $J: \xi \to \xi$ is a complex structure compatible with α (meaning that $d\alpha(J\cdot, J\cdot) = d\alpha$ and $d\alpha(\cdot, J\cdot) > 0$), then we can uniquely extend J to an almost complex structure on the symplectization by declaring that $J\partial_t = R_\alpha$ (and thus $JR_\alpha = -\partial_t$), and J will immediately be compatible with $\omega = d(e^t\alpha)$. From now on we assume J to be fixed.

The J-holomorphic curves we consider will be of the form $u: (\Sigma, j) \to (\mathbb{R} \times M, J)$, with $\Sigma = S^2 \setminus \{y_+, y_1, \ldots, y_s\}$, where $y_+, y_1, \ldots, y_s \in S^2$ are some selected points and j is some complex structure on S^2. Moreover, each puncture in Σ will have an associated *asymptotic marker*. For a puncture p, an asymptotic marker is an element of $(T_pS^2 \setminus \{0\})/\mathbb{R}_+$; in polar coordinates (ρ, θ) about the puncture we may identify an asymptotic marker with a ray $\{\theta = \theta_0\}$.

Specifically, we want to select closed Reeb orbits $\gamma_+, \gamma_1, \ldots, \gamma_s$ of period $T_+, T_1, \ldots, T_s$, respectively, and ask u to be a j-J-holomorphic curve that converges to vertical cylinders over these closed Reeb orbits near the punctures. We can write $u: \Sigma \to \mathbb{R} \times M$ as a pair of maps $a: \Sigma \to \mathbb{R}$ and $\tilde{u}: \Sigma \to M$. This allows us
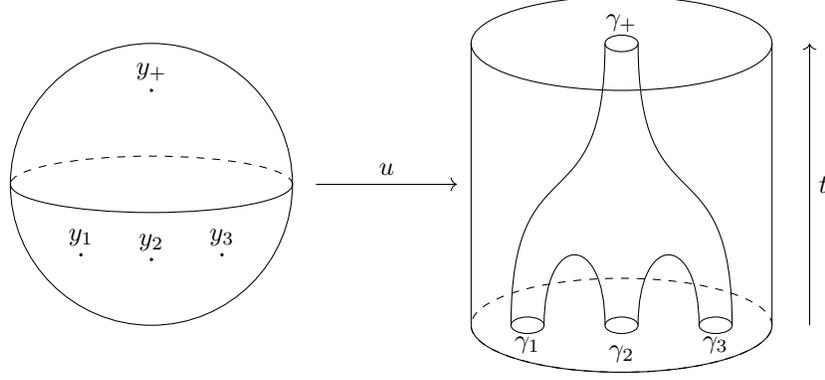
FIGURE 1. A J-holomorphic curve in $\mathcal{M}(\gamma_+; \gamma_1, \gamma_2, \gamma_3)$.

to express the behavior we expect near punctures as limits: With polar coordinates (ρ, θ) near a puncture $p \in S^2$, we want

$$(5) \qquad \lim_{\rho \to 0} a(\rho, \theta) = \begin{cases} +\infty, & p = y_+ \\ -\infty, & p = y_i \end{cases}, \quad \text{and} \quad \lim_{\rho \to 0} \tilde{u}(\rho, \theta) = \begin{cases} \gamma_+(-T_+\theta/2\pi), & p = y_+ \\ \gamma_i(T_i\theta/2\pi), & p = y_i \end{cases}.$$

In addition to requiring u to take the punctures to cylinders over Reeb chords, we also require that u preserves markers. The asymptotic marker associated with each puncture corresponds to a ray $\{\theta = \theta_0\}$ in the local polar coordinates; let $\theta_+, \theta_1, \ldots, \theta_s$ represent these rays, and let $x_+, x_1, \ldots, x_s$ be the marked points selected earlier on the simple orbits underlying $\gamma_+, \gamma_1, \ldots, \gamma_s$. Then we require that

$$(6) \qquad \lim_{\rho \to 0} \tilde{u}(\rho, \theta_+) = x_+ \quad \text{and} \quad \lim_{\rho \to 0} \tilde{u}(\rho, \theta_i) = x_i.$$

Because we defined $J\partial_t = R_\alpha$, the j-J-holomorphicity of u will follow from the conditions in (5): near y_+ we have

$$(du \circ j)(\partial_\rho) = du(\partial_\theta) = -R_\alpha = J(-\partial_t) = (J \circ du)(\partial_\rho)$$

and

$$(du \circ j)(\partial_\theta) = du(-\partial_\rho) = \partial_t = J(-R_\alpha) = (J \circ du)(\partial_\theta),$$

so indeed $du \circ j = J \circ du$. Chasing down the difference in signs gives the holomorphicity near the other punctures. For fixed Reeb orbits $\gamma_+, \gamma_1, \ldots, \gamma_s$, we now define the moduli space

$$\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$$

to be the set of j-J-holomorphic curves $u \colon (\Sigma, j) \to (\mathbb{R} \times M, J)$ satisfying (5) and (6), subject to the following equivalence relation: the curves

$$u \colon S^2 \setminus \{y_+, y_1, \ldots, y_s\} \quad \text{and} \quad u' \colon S^2 \setminus \{y'_+, y'_1, \ldots, y'_s\}$$

with asymptotic markers $r_+, r_1, \ldots, r_s$ and $r'_+, r'_1, \ldots, r'_s$ are equivalent if there is a biholomorphism $h \colon S^2 \to S^2$ so that $h(y_i) = y'_i$ for $i = +, 1, \ldots, s$, $dh(r_i) = r'_i$ for $i = +, 1, \ldots, s$, and $u = u' \circ h$. There is naturally an $\mathbb{R}$-action on $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$ given by translation in the t-direction of the symplectization $\mathbb{R} \times M$.

3.2. **The compactness result.** In the event that $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$ is a compact, oriented 1-dimensional manifold, we can obtain a signed count of $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)/\mathbb{R}$, and we expect such counts to be involved in the differential of our DGA. It is important, then, that we make some compactness and transversality statements about the moduli spaces $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$. Because the symplectization $\mathbb{R} \times M$ does not have bounded geometry at the negative end, Gromov's compactness theorem is not available to us, so we must give a different statement about the niceness of our moduli spaces. Obtaining such results requires a careful perturbation of the Cauchy-Riemann equation defining J-holomorphic curves, and treatments of these ideas can be found in [9], [1], and [5]. We will not explore this here, but under certain restrictive conditions, we obtain the following result.
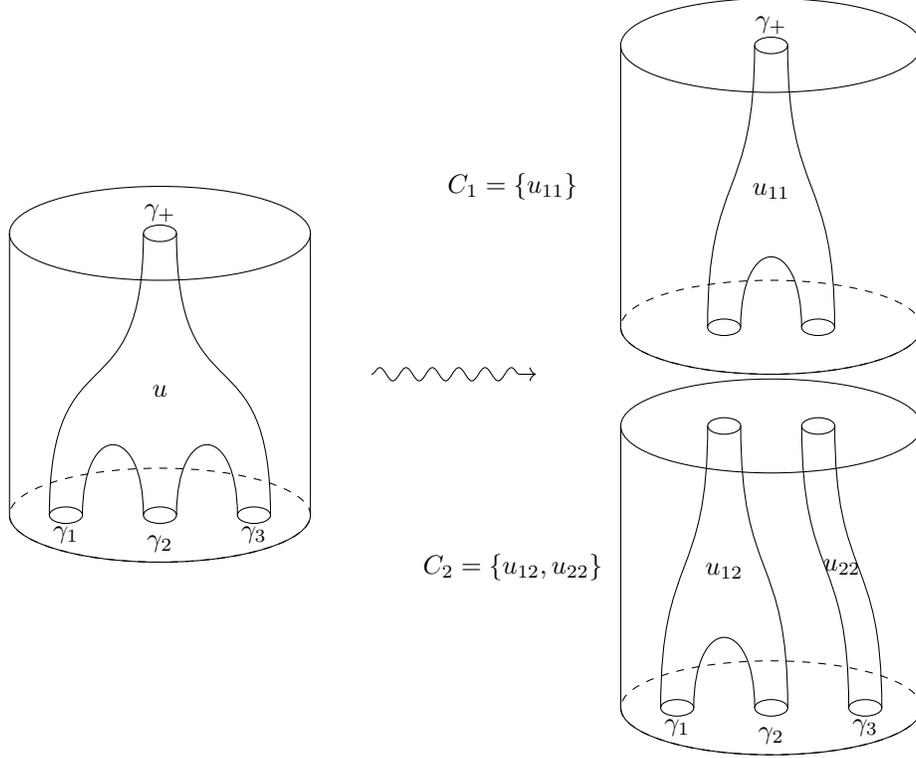
FIGURE 2. A broken J-holomorphic curve in the boundary of $\mathcal{M}(\gamma_+; \gamma_1, \gamma_2, \gamma_3)/\mathbb{R}$.

Theorem 3. *Let* $\gamma_+, \gamma_1, \ldots, \gamma_s \subset M^{2n-1}$ *be closed Reeb orbits. Then* $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)/\mathbb{R}$ *is a union of compact manifolds with corners along a codimension-1 branching locus. Each manifold with corners in this union has dimension*

$$(7) \qquad (n-3)(1-s) + \mu_{CZ}(\gamma_+) - \sum_{i=1}^{s} \mu_{CZ}(\gamma_i) + 2c_1^{rel}(\xi, \Sigma) - 1,$$

where $c_1^{rel}(\xi, \Sigma)$ *is the first Chern class of* ξ *on* Σ, *relative to the fixed trivializations of* ξ *along the closed Reeb orbits at the punctures.*

Repeating a common theme for Floer theories, the boundary of $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$ is made up of broken versions of the curves of interest, such as the broken J-holomorphic curve seen in Figure 2. A *broken J-holomorphic curve* (also called a *holomorphic building of height N*) is a set $C_1, \ldots, C_N$ of finite collections $C_k = \{u_{1k}, \ldots, u_{lk}\}$ of J-holomorphic curves of the type described in section 3.1 such that (1) the positive closed orbits of C_k coincide with the negative closed orbits of C_{k-1} for $k = 2, \ldots, N$, (2) the sole positive closed orbit of C_1 is γ_+, and (3) the negative closed orbits of C_N are $\gamma_1, \ldots, \gamma_s$.

Once the notion of convergence is nailed down for holomorphic buildings (something we will not pursue here), one can obtain the following (see [1] or [4]):

Theorem 4. *Fix distinct points* $y_+, y_1, \ldots, y_s \in S^2$ *and closed Reeb orbits* $\gamma_+, \gamma_1, \ldots, \gamma_s \subset M$, *and let* $\Sigma = S^2 \setminus \{y_+, y_1, \ldots, y_s\}$. *If the sequence of curves*

$$(u_n: (\Sigma, j) \to (\mathbb{R} \times M, J)) \subset \mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$$

has a uniform energy bound $E_0 > 0$, *then there is a subsequence of these curves which converges to a holomorphic building of height* $N \geq 1$.

3.3. **Narrowing the moduli space.** One way to obtain the necessary energy bounds on the moduli spaces $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$ is to specify homology classes for their J-holomorphic curves. For a moduli space such as $\mathcal{M}^A(\gamma_+; \gamma_1, \ldots, \gamma_s)$, with $A \in H_2(M, \mathbb{Z})$, to make sense, we must first assign to each J-holomorphic curve in

$\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$ a homology class in $H_2(M, \mathbb{Z})$.

Given a J-holomorphic curve $u = (a, \tilde{u})\colon \Sigma \to \mathbb{R} \times M$, we will produce an element of $H_2(M, \mathbb{Z})$ by "capping off" the surface $\tilde{u}(\Sigma) \subset M$, which has boundary components corresponding to the closed Reeb orbits $\gamma_+, \gamma_1, \ldots, \gamma_s$. So for each closed Reeb orbit γ_0 in M we must produce a surface S_{γ_0} which will be used to cap off $\tilde{u}(\Sigma)$.

If $[\gamma_0] = 0 \in H_1(M, \mathbb{Z})$, it is natural for us to choose S_{γ_0} to be a spanning surface for γ_0; in case γ_0 is not null-homologous, we do a bit more work. First, write $H_1(M, \mathbb{Z})$ as a direct sum $F \oplus T$ of a free module F and a torsion module T, and choose curves $C_1, \ldots, C_j \subset M$ representing a basis for F and curves $D_1, \ldots, D_l \subset M$ representing a minimal generating set for T. If $[\gamma_0] \in F$, we choose $S_{\gamma_0} \subset M$ providing a homology between γ_0 and a linear combination of the curves $C_1, \ldots, C_j$, while if $[\gamma_0] \notin F$, we choose S_{γ_0} providing a homology between γ_0 and a linear combination of the curves $C_1, \ldots, C_j$ and $D_1, \ldots, D_l$, with minimal nonnegative coefficients of $D_1, \ldots, D_l$. With capping surfaces chosen for all closed Reeb orbits of M, we glue $S_{\gamma_+}, S_{\gamma_1}, \ldots, S_{\gamma_s}$ to $\tilde{u}(\Sigma)$ along its boundary components. Because γ_+ is homologous to $\gamma_1 \sqcup \cdots \sqcup \gamma_s$ (since $\tilde{u}(\Sigma)$ provides a cobordism), the resulting surface is closed. Finally, we let $A \in H_2(M, \mathbb{Z})$ be the homology class of this surface and call A the homology class of u. We denote by $\mathcal{M}^A(\gamma_+; \gamma_1, \ldots, \gamma_s)$ the collection of J-holomorphic curves in $\mathcal{M}(\gamma_+; \gamma_1, \ldots, \gamma_s)$ with homology class A.

The selection of a spanning surface S_{γ_0} for each closed Reeb orbit γ_0 resolves the ambiguity of our grading, and also allows us to simplify the expression (7) for the dimension of the moduli space $\mathcal{M}^A(\gamma_+; \gamma_1, \ldots, \gamma_s)$. We may fix trivializations of ξ over the representative curves $C_1, \ldots, C_j$ and $D_1, \ldots, D_l$ and use the spanning surfaces to extend these trivializations to a trivialization over γ_0. These preferred trivializations then provide the grading we use in contact homology[2]. In case $\gamma_0 \notin F$, choose $k \in \mathbb{N}$ minimal with $\gamma_0^k \in F$. Then we have two different trivializations of ξ over γ_0^k: one obtained as above, using $S_{\gamma_0^k}$, and another by pulling back the trivialization over γ_0. We then have

$$|\gamma_0| = \mu_{CZ}(\gamma_0) - 2\frac{w}{k} + n - 3 \in \frac{1}{k}\mathbb{Z},$$

where w is the rotation number of the pullback trivialization of ξ over γ_0^k with respect to the spanning-surface trivialization. So we have

$$\mu_{CZ}(\gamma_0) = |\gamma_0| - (n-3) + 2\frac{w}{k}.$$

Substituting into (7) yields

(8) $$\dim \mathcal{M}^A(\gamma_+; \gamma_1, \ldots, \gamma_s) = |\gamma_+| - \sum_{i=1}^{s} |\gamma_i| + 2\langle c_1(\xi), A\rangle.$$

(Recall that (7) gives $\dim \mathcal{M}^A(\gamma_+; \gamma_1, \ldots, \gamma_s)/\mathbb{R}$.) This expression will allow us to easily check that our differential has a grading of -1.

3.4. **The differential.** Recall the differential in Morse homology: the differential of a critical point of index k is a weighted sum of the critical points with index $k-1$, and the weights are intended to count the number of curves from one critical point to the next. Our differential follows this pattern, and we begin by defining the relevant weights. Let Γ_- be a collection of good Reeb orbits:

$$\Gamma_- = (\gamma_1^{i_1}, \ldots, \gamma_s^{i_s}).$$

Here the notation $\gamma_j^{i_j}$ does not represent the i_j-fold cover of the Reeb orbit γ_j, but instead indicates that Γ_- contains i_j copies of γ_j. The collection Γ_- will provide the negative ends for a moduli space of J-holomorphic curves, and we do not want to distinguish between curves which agree up to, say, having switched a pair of copies of γ_1. To this end, we define the *multiplicity* of Γ_- to be

$$m_{\Gamma_-} = \prod_{j=1}^{s} i_j! m(\gamma_j)^{i_j},$$

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[2]The trivialization used in the above example cannot be extended over a spanning surface, and is therefore *not* the preferred trivialization

where $m(\gamma_j)$ is the multiplicity of γ_j over a simple closed Reeb orbit. For a homology class $A \in H_2(M, \mathbb{Z})$ and a good Reeb orbit γ_+, we define

$$n^A_{\gamma_+, \Gamma_-} = \frac{\#(\mathcal{M}^A(\gamma_+; \Gamma_-)/\mathbb{R})}{m_{\Gamma_-}}$$

if $\dim \mathcal{M}^A(\gamma_+; \Gamma_-) = 1$, and define $n^A_{\gamma_+, \Gamma_-} = 0$ otherwise. These numbers give a (signed[3]) count of the J-holomorphic curves with a positive end at γ_+ and negative ends at Γ_-, with homology class $A \in H_2(M, \mathbb{Z})$. Because the spaces $\mathcal{M}^A(\gamma_+; \Gamma_-)/\mathbb{R}$ are compact, the numbers $n^A_{\gamma_+, \Gamma_-}$ are finite, and are zero for almost all $A \in H_2(M, \mathbb{Z})$. For this reason we may define the coefficients

$$n_{\gamma_+, \Gamma_-} = \sum_{A \in H_2(M,\mathbb{Z})} n^A_{\gamma_+, \Gamma_-} e^A \in \mathbb{Q}[H_2(M, \mathbb{Z})].$$

At long last, we define the differential on generators of A by

(9) $$d\gamma_+ := \sum_{\Gamma_-} n_{\gamma_+, \Gamma_-} \gamma_1^{i_1} \cdots \gamma_s^{i_s}.$$

The differential is extended to all of A using the graded Leibniz rule.

To verify that the differential has degree -1, suppose $n^A_{\gamma_+, \Gamma_-} \neq 0$, so that (8) gives

$$1 = |\gamma_+| - \sum_{j=1}^{s} i_j |\gamma_j| + 2\langle c_1(\xi), A \rangle.$$

Recall the grading on $\mathbb{Q}[H_2(M, \mathbb{Z})]$:

$$|n^A_{\gamma_+, \Gamma_-} e^A| = -2\langle c_1(\xi), A \rangle.$$

From this we see that

$$|n^A_{\gamma_+, \Gamma_-} e^A \gamma_1^{i_1} \cdots \gamma_s^{i_s}| = |n^A_{\gamma_+, \Gamma_-} e^A| + \sum_{j=1}^{s} i_j |\gamma_j| = -2\langle c_1(\xi), A \rangle + \sum_{j=1}^{s} i_j |\gamma_j| = |\gamma_+| - 1,$$

so indeed $|d\gamma_+| = |\gamma_+| - 1$.

3.5. **Contact homology.** Finally we define the contact homology of the contact manifold (M, ξ). Our ability to make this definition follows from the following theorem.

Theorem 5. *(Eliashberg-Givental-Hofer, [7]) Let (M, ξ) be a co-orientable, compact contact manifold satisfying the various assumptions made during this talk[4]. Then the pair (A, d) defined above is a differential graded algebra, and the homology $H_*(\mathsf{A}, d)$ is independent of the contact form α chosen with $\ker \alpha = \xi$, of the complex structure J on ξ, and of the perturbation of the moduli spaces.*

Definition. The *contact homology* $HC_*(M, \xi)$ is the homology $H_*(\mathsf{A}, d)$ of the DGA described above.

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[3]Obtaining a signed count of course requires the moduli spaces to be oriented. For information on how to produce coherent orientations on the moduli spaces (and why these orientations require the exclusion of bad Reeb orbits), see [5].

[4]Whatever those may be.

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