

Quantum cohomology and the Dubrovin connection
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This talk is meant to provide background for Prof. Morgan's talk on Friday. References for this material include McDuff-Salamon's *J-holomorphic Curves and Quantum Cohomology*, [Givental's notes](#), and Prof. Morgan.

1. MOTIVATION

- **Notation.** For any smooth manifold M we'll use $H_*(M)$ and $H^*(M)$ to denote the free parts of $H_*(M; \mathbb{Z})$ and $H^*(M; \mathbb{Z})$, respectively. By this we mean that $H_*(M) = H_*(M; \mathbb{Z}) / \text{Tor}(H_*(M; \mathbb{Z}))$. We will write (X, ω) for a closed, symplectic $2n$ -manifold.
- Our goal today is to define *quantum cohomology*, and in particular the quantum cup product. We will also show how the quantum cup product gives rise to a flat connection on a trivial bundle over $\mathbb{C}^\times \times H^2(X; \mathbb{C})$.
- Consider the usual cup product on cohomology. Because of the assumed isomorphism $H^k(M) \cong \text{Hom}(H_k(M), \mathbb{Z})$, this can be defined by its *structural constants*. To be precise, we can define the cup product $a \smile b \in H^{j+k}(M)$ of cohomology classes $a \in H^j(M)$ and $b \in H^k(M)$ to be the unique class satisfying

$$\langle a \smile b, c \rangle = \text{PD}[a] \cdot \text{PD}[b] \cdot \text{PD}[c]$$

for each $c \in H^{n-(j+k)}(M)$, where $n = \dim M$.

- Today we want to deform the cup product. As in the case just considered, we will define the cup product by its structural constants. Namely, we will replace the count of intersections between homology classes with a count of J -holomorphic curves passing through given classes.

2. GROMOV-WITTEN INVARIANTS

- As stated above, we will define the quantum cup product by defining its structural constants. The structural constants in the usual cup product are determined by the intersection pairing, which provides a signed count of the intersections between generic cycles representing some homology classes $\text{PD}[a_1], \dots, \text{PD}[a_k]$. For the quantum cup product we replace the intersection counts with *Gromov-Witten invariants*, which instead count pseudoholomorphic spheres that meet the cycles at prescribed points.
- Let (X, ω, J) be a closed symplectic $2n$ -manifold with compatible almost complex structure J , and fix a homology class $A \in H_2(X)$. We assume that J is chosen so that the moduli space of J -holomorphic A -curves $u: \mathbb{C}\mathbb{P}^1 \rightarrow X$ is a smooth manifold.
- For any $k \geq 1$, we consider the moduli space of J -holomorphic A -curves equipped with k marked points. By adding stable J -holomorphic maps and nodal marked curves this space can be compactified, and we denote this compactification by $X_{k,A}$. This space admits a fundamental cycle

$$[X_{k,A}] \in H_d(X_{k,A}; \mathbb{Q}),$$

where $d = 2c_1(A) + 2n + 2(k - 3)$, and for $1 \leq i \leq k$ we may define

$$\text{ev}_i: X_{k,A} \rightarrow X$$

by evaluating each curve in $X_{k,A}$ at its i th marked point.

- Now take classes (of pure degree)

$$a_1, \dots, a_k \in H^*(X; \mathbb{Z}).$$

We want to use the curves in $X_{k,A}$ to generalize the intersection number $\text{PD}[a_1] \cdots \text{PD}[a_k]$. First, we use the evaluation maps ev_i to construct the cohomology class

$$a := \text{ev}_1^* a_1 \smile \cdots \smile \text{ev}_k^* a_k \in H^*(X_{k,A}; \mathbb{Q}),$$

a class of degree

$$\deg(a) = \sum_{i=1}^k \deg(a_i).$$

If $\deg(a) = d$, then taking the cap product with the fundamental cycle $[X_{k,A}]$ gives an element of $H_0(X_{k,A}; \mathbb{Q}) = \mathbb{Q}$. For this reason we define

$$\langle a_1, \dots, a_k \rangle_A := [X_{k,A}] \frown a = \int_{[X_{k,A}]} a \in \mathbb{Q}$$

whenever $\deg(a) = d$, and define $\langle a_1, \dots, a_k \rangle_A$ to be zero whenever $\deg(a) \neq d$. The numbers are the *Gromov-Witten invariants* of (X, ω, J) .

- How should we interpret these numbers? Suppose we have cycles $\alpha_1, \dots, \alpha_k$ in X representing the Poincaré duals of a_1, \dots, a_k . We can think of $\langle a_1, \dots, a_k \rangle_A$ as giving a signed intersection count of the fundamental $[X_{k,A}]$ with these cycles¹. Then a positive contribution to this count is given by an A -curve u in $X_{k,A}$ which meets α_i at $u(z_i)$, for $i = 1, \dots, k$. That is, instead of counting the number of points where all of the cycles $\alpha_1, \dots, \alpha_k$ meet, we count the number of A -curves which meet all of these cycles simultaneously, with each intersection occurring at a specified point in the domain of u . So the Gromov-Witten invariants serve as some sort of *fuzzy* intersection numbers. Notice that setting $A = 0 \in H_2(X; \mathbb{Z})$ recovers the usual intersection number.
- **Note.** We should point out that these invariants do not strictly depend on the symplectic and almost complex structures used to define them. Namely, we may deform either of these structures without changing the invariants.
- One important property of the Gromov-Witten invariants is the *divisor property*, which says that for any $p \in H^2(X)$ and any cohomology classes $a_1, \dots, a_k \in H_*(M)$, we have

$$\langle a_1, \dots, a_k, p \rangle_A = p(A) \cdot \langle a_1, \dots, a_k \rangle_A,$$

for all $A \in H_2(X)$. (Notice that the dimension conditions of the two sides cooperate.) This property is morally correct: the number $p(A)$ should count the intersections between A and the divisor $\text{PD}[p]$, and the curve-count on the left should reduce to this intersection count multiplied by the curve-count on the right.

- **Example.** Let's think about a toy example. Let $X = \mathbb{C}\mathbb{P}^n$, with the usual complex structure and symplectic form. We will trust that the transversality conditions needed to define the Gromov-Witten invariants are met. Now let $p = \text{PD}[\mathbb{C}\mathbb{P}^{n-1}] \in H^2(X)$ be the usual generator of $H^2(X)$. Notice that this is algebraically dual to the class $L = [\mathbb{C}\mathbb{P}^1] \in H_2(X)$, and that $\text{PD}[p^n] = \text{pt}$, the homology class of a point. Now $\deg(p^n) = 2n$ and $c_1(L) = n + 1$, so

$$\deg(p^n) + \deg(p^n) = 4n = 2n + 2c_1(L) + 2(2 - 3).$$

Namely, the number $\langle p^n, p^n \rangle_L$ is well-defined. Unsurprisingly,

$$\langle p^n, p^n \rangle_L = 1,$$

representing the fact that there's a unique line between any two distinct points in $\mathbb{C}\mathbb{P}^n$. Moreover, since $p(L) = 1$, we see that

$$\langle p^n, p^n, p \rangle_L = p(L) \langle p^n, p^n \rangle_L = 1,$$

because the unique line through our two points must intersect the divisor $\mathbb{C}\mathbb{P}^{n-1}$ once.

¹To make this precise, maybe we should consider the product $\alpha_1 \times \dots \times \alpha_k$ in $X \times \dots \times X$ and count its intersections with $\text{ev}_1([X_{k,A}]) \times \dots \times \text{ev}_k([X_{k,A}])$.

3. QUANTUM COHOMOLOGY

- To begin, choose a basis $\{A_1, \dots, A_N\}$ for $H_2(X)$, and let $\{q_1, \dots, q_N\}$ be the corresponding coordinates on $H^2(X; \mathbb{C})$. That is, $q_i: H^2(X; \mathbb{C}) \rightarrow \mathbb{C}$ is defined by $q_i(A_j^*) = \delta_{ij}$, where $A_j^* \in H^2(X) \cong \text{Hom}(H_2(X), \mathbb{Z})$ is the algebraic dual of A_j . The *quantum cohomology ring* is

$$QH^*(X) := H^*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, \dots, q_N]].$$

We determine a grading on $QH^*(X)$ by defining $\deg(q_i) = 2c_1(A_i)$, so that an element of pure degree k in $QH^*(X)$ may be written

$$\sum_{i=1}^N a^i q_i \in QH^k(X),$$

for some cohomology classes $a^i \in H^{k-2c_1(A_i)}(X; \mathbb{Q})$.

- We will define the quantum cup product for elements of $H^*(X; \mathbb{Q})$ and then extend linearly to $H^*(X; \mathbb{Q})$. Before doing so, we fix notation. Given $A \in H_2(X)$, we may write $A = \sum_i \alpha^i A_i$ for some $\alpha^i \in \mathbb{Z}$. We then define $q^A := \prod_i q_i^{\alpha^i}$. Now let $\langle \cdot, \cdot \rangle$ be the Poincaré duality pairing on cohomology. Given

$$a \in H^j(X; \mathbb{Q}), \quad b \in H^k(X; \mathbb{Q}), \quad \text{and} \quad c \in H^\ell(X; \mathbb{Q}),$$

we define

$$\langle a \circ b, c \rangle := \sum_{A \in H_2(X)} \langle a, b, c \rangle_A q^A \in \mathbb{Q}[[q_1, \dots, q_N]].$$

Notice that because $\langle a, b, c \rangle_A$ is only nonzero when $\deg(a) + \deg(b) + \deg(c) = 2c_1(A) + 2n$, the pairing $\langle a \circ b, c \rangle$ vanishes unless $\deg(a) + \deg(b) + \deg(c) \equiv \text{mod } 2N$, where N is the *minimal Chern number* of X .

- For the sake of computing $a \circ b$, it will help to write

$$a \circ b = \sum_{A \in H_2(X)} (a \circ b)_A q^A,$$

with $(a \circ b)_A \in H^{j+k-2c_1(A)}(X; \mathbb{Q})$ defined by

$$\langle (a \circ b)_A, c \rangle = \langle a, b, c \rangle_A, \quad \text{for all } c \in H^{2n-(j+k-2c_1(A))}(X; \mathbb{Q}).$$

- What is the constant term in this expansion? That is, what happens when we set $q = 0$? If $\deg(q^A) = 0$, then $c_1(A) = 0$. Assume for the moment that (X, ω) is monotone, so that any class $A \in H_2(X)$ with $c_1(A) = 0$ also has $\omega(A) = 0$. Notice that for any non-constant J -holomorphic curve u , ω restricts to an area form on $\text{Im } u$. In particular, $\omega([\text{Im } u]) \neq 0$. So the curves with $\omega([\text{Im } u]) = 0$ must be constant, meaning that

$$\langle a, b, c \rangle_0 = \text{PD}[a] \cdot \text{PD}[b] \cdot \text{PD}[c].$$

So $(a \circ b)_0 = a \smile b$, and hence we have a *deformed cup product*.

- Some properties of the quantum cohomology:
 - (1) it's (super) commutative: $a \circ b = (-1)^{\deg(a) \deg(b)} b \circ a$;
 - (2) it's unital: the canonical generator $\mathbf{1} \in H^0(X)$ is the unit element;
 - (3) it's associative: $(a \circ b) \circ c = a \circ (b \circ c)$;
 - (4) it has the Frobenius property: $\langle a \circ b, c \rangle = \langle a, b \circ c \rangle$.

These properties make quantum cohomology a *commutative Frobenius algebra*. The associativity is not at all obvious, so we'll dwell on it.

- **Example.** Consider $X = \mathbb{C}P^n$ with its usual structures. A basis for $H_2(X; \mathbb{Q})$ is given by the single class $[\mathbb{C}P^1]$, whose dual in $H^2(X; \mathbb{Q})$ we write as q . Then

$$QH^*(\mathbb{C}P^n) = H^*(\mathbb{C}P^n; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q]].$$

Now $H^*(\mathbb{C}\mathbb{P}^n) = \mathbb{Q}[p]$, where $p \in H^2(\mathbb{C}\mathbb{P}^n)$ is the usual generator. We're interested in computing quantum cup products $p^k \circ p^\ell$. For instance, consider the case $k = 1, \ell = n$. We may write

$$p \circ p^n = \sum_{m=0}^{\infty} (p \circ p^n)_{mL} q^m,$$

where $L = [\mathbb{C}\mathbb{P}^1]$. Now $(p \circ p^n)_{mL}$ is a cohomology class of degree

$$2 + 2n - 2mc_1(L) = 2 + 2n - 2m(n+1) = (2n+2)(1-m),$$

and thus will only be nontrivial in case $m = 1$. So in fact

$$p \circ p^n = (p \circ p^n)_L q.$$

The class $(p \circ p^n)_L \in H^0(X; \mathbb{Q})$ is determined by its pairing with $\text{PD}[\text{pt}]$, and we see that

$$\langle (p \circ p^n)_L, \text{PD}[\text{pt}] \rangle = \langle p, p^n, \text{PD}[\text{pt}] \rangle_L = \langle \text{PD}[\mathbb{C}\mathbb{P}^{n-1}], \text{PD}[\text{pt}], \text{PD}[\text{pt}] \rangle = 1.$$

So we have $p \circ p^n = q$. More generally, $p^k \circ p^\ell = p^{k+\ell-(n+1)}q$, so

$$QH^*(\mathbb{C}\mathbb{P}^n) \cong \mathbb{Q}[p, q]/(p^{n+1} - q).$$

Notice that setting $q = 0$ recovers $H^*(\mathbb{C}\mathbb{P}^n)$.

4. ASSOCIATIVITY

In this section we will sketch a proof of the associativity of the quantum cup product. McDuff and Salamon credit this proof to Ruan and Tian.

- First, we define a triple product

$$*: QH^j(X) \otimes QH^k(X) \otimes QH^\ell(X) \rightarrow QH^{j+k+\ell}(X).$$

We define this product for $a \in H^j(X)$, $b \in H^k(X)$, and $c \in H^\ell(X)$ and extend linearly to $QH^*(X)$. We have

$$a * b * c = \sum_A (a * b * c)_A q^A \in QH^{j+k+\ell}(X),$$

where the class $(a * b * c)_A \in H^{j+k+\ell-2c_1(A)}(X; \mathbb{Q})$ is defined by

$$\langle (a * b * c)_A, d \rangle = \langle a, b, c, d \rangle_A \quad \text{for all } d \in H^{2n+2c_1(A)-(j+k+\ell)}(X).$$

Notice that the degree conditions on either side of this equation agree.

- **Lemma.** $(a \circ b) \circ c = a * b * c$.

Because the triple product $*$ is obviously skew-symmetric, this lemma will imply associativity. We will prove the lemma at the level of structural constants. That is, we will show that for any $A \in H_2(X)$ and $d \in H^{2n+2c_1(A)-(j+k+\ell)}(X)$,

$$\langle ((a \circ b) \circ c)_A, d \rangle = \langle (a * b * c)_A, d \rangle = \langle a, b, c, d \rangle_A.$$

On the right we have a count of A -curves meeting the four cycles $\text{PD}[a], \text{PD}[b], \text{PD}[c], \text{PD}[d]$ at a 4-tuple of prescribed points. We investigate the quantity on the left.

- First, recall that we may write

$$a \circ b = \sum_B (a \circ b)_B q^B \in QH^{j+k}(X),$$

with $(a \circ b)_B \in H^{j+k-2c_1(B)}(X; \mathbb{Q})$. The defining equation of the quantum cup product suggests that we write

$$\langle ((a \circ b) \circ c)_A, d \rangle = \langle (a \circ b), c, d \rangle_A$$

for any $d \in H^{2n+2c_1(A)-(j+k+\ell)}(X)$, but of course $(a \circ b)$ is an element of $QH^*(X)$ rather than $H^*(X)$. So instead we have

$$\langle ((a \circ b) \circ c)_A, d \rangle = \sum_B \langle (a \circ b)_{A-B}, c, d \rangle_B,$$

with the shift $A - B$ needed to make degrees match. Consider the curves counted by $\langle (a \circ b)_{A-B}, c, d \rangle_B$. Such a curve passes through $\text{PD}[c]$ and $\text{PD}[d]$, and also intersects $\text{PD}[(a \circ b)_{A-B}]$ at some point z . So there is a curve $u \in \mathcal{M}(A - B, J)$ passing through $\text{PD}[a]$, $\text{PD}[b]$ and z . That is, $\langle (a \circ b)_{A-B}, c, d \rangle_B$ counts the $(A - B, B)$ -cusp curves passing through the Poincaré duals α, β, γ , and δ of a, b, c , and d .

- The big idea for proving associativity is then this: counting A -curves passing through $\alpha, \beta, \gamma, \delta$ is the same as counting $(A - B, B)$ -cusp curves through these four classes and summing over all $B \in H_2(X)$.
 (1) **Why does this get us associativity?** In symbols, the big idea says that

$$\langle a, b, c, d \rangle_A = \sum_B \langle (a \circ b)_{A-B}, c, d \rangle_B.$$

But this means precisely that

$$\langle (a * b * c)_A, d \rangle = \langle ((a \circ b) \circ c)_A, d \rangle$$

for each relevant $A \in H_2(X)$ and $d \in H^*(X)$, so $(a \circ b) \circ c = a * b * c$, as desired.

- (2) **Why should this be true?** We want to show that counting A -curves is the same as counting $(A - B, B)$ -cusp curves. Let $\mathbf{z} = (\infty, 1, 0, z)$ for some $z \in \mathbb{C} - \{0, 1\}$ and define the obvious map $e_{\mathbf{z}}: X_{4,A} \rightarrow X \times X \times X \times X$. Morally, the count of A -curves is given by the intersection count

$$\langle a, b, c, d \rangle_A = e_{\mathbf{z}} \cdot (\alpha \times \beta \times \gamma \times \delta),$$

and this count is independent of z . Now consider sequences $w_\nu \in X_{4,A}$ and $z_\nu \in \mathbb{C}^{\mathbb{P}^1}$, with $w_\nu \in e_{\mathbf{z}}^{-1}(\alpha \times \beta \times \gamma \times \delta)$ and $z_\nu \rightarrow 0$. There are two possibilities:

- (a) The sequences $w_\nu(z_\nu)$ and $w_\nu(0)$ converge to the same point. In this case, the common point must lie in $\gamma \cap \delta$, so w_ν converges to a $(A - B, B)$ -curve, with $B = 0$.
- (b) These sequences converge to different points. In this case, the derivative of w_ν blows up at some point near 0, giving us a bubble in the limit. Away from this bubble we have $w_\infty \in X_{3,A-B}$ with $w_\infty(\infty) \in \alpha$ and $w_\infty(1) \in \beta$, where B is the homology class of the bubble.

In either case we see that when z is sufficiently near 0, each A -curve determines an $(A - B, B)$ -cusp curve for some B , so the two counts agree.

5. THE DUBROVIN CONNECTION

The purpose of this last section is to repackage the quantum cup product as a connection on a trivial vector bundle over $\mathbb{C}^\times H^2(X; \mathbb{C})$. The Frobenius algebra structure of $QH^*(X)$ will make this a flat connection, and on Friday we'll learn about situations where this connection is mirrored by a Gauss-Manin connection.

- To begin, we write

$$H^{2*}(X; \mathbb{C}) = \bigoplus_{i \in \mathbb{Z}} H^{2i}(X; \mathbb{C}),$$

a complex vector space. Notice that the Poincaré duality pairing and quantum cup product can be extended to $H^{2*}(X; \mathbb{C})$: for $a, b \in H^{2*}(X; \mathbb{C})$ we define

$$\langle a, b \rangle = \int_X \bar{a} \smile b,$$

and then define $a \circ b$ via structural constants as before.

- We let $\mathcal{H} \rightarrow H^2(X; \mathbb{C})$ denote the trivial bundle with fiber $H^{2*}(M; \mathbb{C})$. For some fixed $z \in \mathbb{C}^\times$, the quantum cup product is encoded in the form of a connection on \mathcal{H} , defined as follows. Let p^1, \dots, p^N be the basis of $H^2(X)$ algebraically dual to the basis A_1, \dots, A_N of $H_2(X)$. That is, $p^i \in H^2(X)$ is defined by $p^i(A_j) = \delta_{ij}$. Define

$$\nabla = d + \frac{1}{z} \sum_{i=1}^N (p^i \circ) d(\ln q_i).$$

This is the *Dubrovin connection*, and the commutativity and associativity of the quantum cup product tell us that it is flat.

Proposition 5.1. *The Dubrovin connection is flat, for any $z \in \mathbb{C}^\times$.*

Proof. The connection form is given by

$$A^1 = \frac{1}{z} \sum_{i=1}^N (p^i \circ) d(\ln q_i),$$

so it will suffice to show that the 2-forms $zA^1 \wedge zA^1$ and $d(zA^1)$ vanish. For the first form we simply notice that

$$zA^1 \wedge zA^1 = \frac{1}{2} \left[\sum_{i=1}^N (p^i \circ) d(\ln q_i), \sum_{j=1}^N (p^j \circ) d(\ln q_j) \right],$$

and the commutator on the right will vanish by the associativity and commutativity of the quantum cup product.

To show that $d(zA^1) = 0$, we first establish the following differential equation: for any $1 \leq i, j \leq N$ and any $a, b \in H^*(X; \mathbb{C})$ of appropriate degree,

$$(1) \quad q_i \frac{\partial}{\partial q_i} (\langle a, p^j \circ b \rangle) = q_j \frac{\partial}{\partial q_j} (\langle a, p^i \circ b \rangle).$$

To see that this is so, define numbers $\alpha^i = p^i(A)$ for each $A \in H_2(X)$ and notice that

$$\begin{aligned} q_i \frac{\partial}{\partial q_i} (\langle a, p^j \circ b \rangle) &= q_i \frac{\partial}{\partial q_i} \left(\sum_A \langle a, p^j \circ b \rangle_A q^A \right) \\ &= \sum_A q_i \langle a, p^j \circ b \rangle_A \frac{\partial}{\partial q_i} (q^A) \\ &= \sum_A \langle a, b \rangle_A \alpha^i \alpha^j, \end{aligned}$$

with the last equality using both the divisor property of the Gromov-Witten invariants and the fact that $q_i (\partial / \partial q_i)(q^A) = \alpha^i q^A$. Next we observe that for each $1 \leq j \leq N$,

$$zA^1(\partial_{q_j}) = \sum_{i=1}^N (p^i \circ) \left(\frac{dq^i}{q_i} \left(\frac{\partial}{\partial q_j} \right) \right) = \frac{(p^j \circ)}{q_j}.$$

So

$$2q_i q_j d(zA^1)(\partial_{q_i}, \partial_{q_j}) = q_i \frac{\partial}{\partial q_i} ((p^j \circ)) - q_j \frac{\partial}{\partial q_j} ((p^i \circ)),$$

since $[\partial_{q_i}, \partial_{q_j}] = 0$. But (1) tells us that this last expression will vanish, so the Dubrovin connection is indeed flat. \square

- The Dubrovin connection can naturally be extended to the trivial $H^{2*}(X; \mathbb{C})$ -connection over $\mathbb{C}^\times \times H^2(X; \mathbb{C})$ by defining

$$\nabla_{\partial_z} = \frac{\partial}{\partial z} + \frac{1}{z} \mu - \frac{1}{z^2} c_1(X) \circ,$$

where $\mu: H^{2*}(X; \mathbb{C}) \rightarrow H^{2*}(X; \mathbb{C})$ is the Hodge grading operator, defined by

$$\mu(a) = \left(\frac{\deg(a) - n}{2} \right) a.$$

Proposition 5.2. *The extended Dubrovin connection is flat.*

Proof. Flatness means that for any vector fields $\partial_{q_i}, \partial_{q_j}$ on $H^2(X; \mathbb{C})$ we have

$$[\nabla_{\partial_{q_i}}, \nabla_{\partial_{q_j}}] = \nabla_{[\partial_{q_j}, \partial_{q_i}]}, \quad [\nabla_{\partial_{q_i}}, \nabla_{\partial_z}] = \nabla_{[\partial_{q_i}, \partial_z]}, \quad \text{and} \quad [\nabla_{\partial_z}, \nabla_{\partial_z}] = \nabla_{[\partial_z, \partial_z]}.$$

We've already established the first equation, and the third is free, so we need only verify the middle equation. To this end, we compute

$$\begin{aligned} [\nabla_{\partial q_i}, \nabla_{\partial z}] &= \left[\frac{\partial}{\partial q_i} + \frac{(p^i \circ)}{z q_i}, \frac{\partial}{\partial z} + \frac{1}{z} \mu - \frac{1}{z^2} c_1(X) \circ \right] \\ &= \left[\frac{(p^i \circ)}{z q_i}, \frac{1}{z} \mu \right] + \left[\frac{(p^i \circ)}{z q_i}, \frac{\partial}{\partial z} \right] - \left[\frac{\partial}{\partial q_i}, \frac{1}{z^2} c_1(X) \circ \right]. \end{aligned}$$

Notice that

$$\left[\frac{(p^i \circ)}{z q_i}, \frac{1}{z^2} c_1(X) \circ \right] = 0$$

by the associativity and commutativity of quantum cup product. We begin computing the three remaining terms. First,

$$\left[\frac{(p^i \circ)}{z q_i}, \frac{\partial}{\partial z} \right] = -\frac{\partial}{\partial z} \left(\frac{(p^i \circ)}{z q_i} \right) = \frac{(p^i \circ)}{z^2 q_i}.$$

Next we see that

$$\begin{aligned} \left[\frac{(p^i \circ)}{z q_i}, \frac{1}{z} \mu \right] &= \frac{1}{z^2 q_i} \left(\sum_A (p^i \circ \mu(-))_A q^A - \mu \left(\sum_A (p^i \circ)_A q^A \right) \right) \\ &= \frac{1}{z^2 q_i} \sum_A (c_1(A) - 1) (p^i \circ)_A q^A \\ &= \frac{1}{z^2 q_i} \sum_A c_1(A) (p^i \circ)_A q^A - \frac{(p^i \circ)}{z^2 q_i}. \end{aligned}$$

So it remains to check that

$$0 = \frac{1}{z^2 q_i} \sum_A c_1(A) (p^i \circ)_A q^A - \frac{(p^i \circ)}{z^2 q_i} + \frac{(p^i \circ)}{z^2 q_i} - \left[\frac{\partial}{\partial q_i}, \frac{1}{z^2} c_1(X) \circ \right].$$

That is, we want to show that

$$\left[\frac{\partial}{\partial q_i}, \frac{1}{z^2} c_1(X) \circ \right] = \frac{1}{z^2 q_i} \sum_A c_1(A) (p^i \circ)_A q^A.$$

To this end, note that because p_1, \dots, p_N is the basis for $H^2(X)$ dual to the basis A_1, \dots, A_N of $H_2(X)$, we may write

$$c_1(X) \circ = \sum_{j=1}^N c_1(A_j) (p^j \circ).$$

Now differentiating gives

$$\frac{\partial}{\partial q_i} (c_1(X) \circ) = \sum_{j=1}^N c_1(A_j) \frac{\partial}{\partial q_i} (p^j \circ) = \sum_{j=1}^N c_1(A_j) \frac{q_j}{q_i} \frac{\partial}{\partial q_j} (p^i \circ),$$

with the second equality coming from (1). We saw above that

$$q_j \frac{\partial}{\partial q_j} (p^i \circ) = \sum_A p^j(A) (p^i \circ)_A q^A,$$

and since $A = \sum_j p^j(A) A_j$, we know that $c_1(A) = \sum_j p^j(A) c_1(A_j)$. So

$$\frac{\partial}{\partial q_i} (c_1(X) \circ) = \frac{1}{q_i} \sum_{j=1}^N \sum_A c_1(A_j) p^j(A) (p^i \circ)_A q^A = \frac{1}{q_i} \sum_A c_1(A) (p^i \circ)_A q^A,$$

as desired. \square