Flexibility in Contact Topology

UCLA Student Topology Seminar
Notes by Austin Christian

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Abstract

These are lecture notes from a working seminar on notions of flexibility and overtwistedness in (mostly higher-dimensional) contact topology. The first half of the seminar was spent studying the classification of overtwisted contact structures in all dimensions, as in [BEM15]. In the second half of the seminar, the relationships between overtwistedness and geometric phenomena such as loose Legendrians, plastikstufe, and contact $(+1)$-surgeries were studied, with heavy reliance on [CMP19].

Contents

Note. The talks were given by the participants of the seminar, who deserve credit for their organization and presentation. Any errors introduced are my own (and I would be glad to hear about them!).

Also, talk 9 is missing. Perhaps I will type up my notes for that talk at some point.

1 An overview of flexibility in contact topology

The ordinary differential equation

$$y^2 + (y')^2 = -4$$

has no real-valued solutions. This claim is obvious when we consider the algebraic, or *formal* problem of finding real-valued functions u, y which satisfy

$$y^2 + u^2 = -4.$$

Because no solutions to this problem exist, there are certainly no solutions which additionally satisfy $u = y'$.

It seems reasonable that we could obstruct solutions to partial differential equations or partial differential relations by showing that a corresponding formal problem cannot be solved. But a truly interesting phenomenon would be the converse: are there partial differential relations for which our only obstacle is the formal problem? Amazingly, our seminar this quarter will provide a number of situations where the answer is *yes*. Whenever such a circumstance exists, we say that we have an *h-principle* for the problem in question, and we think of the problem as being *flexible*.

1.1 Definitions

Flexible problems exist throughout geometry, but the interplay between rigidity and flexibility is especially rich in symplectic and contact geometry. We'll now provide a smattering of basic definitions and results from these disciplines.

Definition. A **symplectic form** on a smooth manifold is a closed, non-degenerate differential 2-form. That is, ω on W^{2n} is symplectic if $d\omega = 0$ and $\omega^n \neq 0$. In this case, we call the pair (W, ω) a *symplectic manifold*.

Definition. Let M^{2n+1} be an oriented smooth manifold. A **contact structure** on M is a maximally non-integrable hyperplane field $\xi \subset TM$. That is, for each point $p \in M$ there is a neighborhood on which we may write $\xi = \ker \alpha$ for some 1-form α satisfying

$$\alpha \wedge (d\alpha)^n > 0. \tag{1.1}$$

We call α a **contact form** for ξ, call (1.1) the **contact condition**, and call the pair (M, ξ) a **contact manifold**.

Remark. (a) We will in fact always assume that ξ is *co-orientable*, meaning that the contact form α satisfies $\xi = \ker \alpha$ on all of M.

(b) If this is your first encounter with contact manifolds, it might be worth checking that

(1) The distribution ξ is co-orientable if and only if $TM/\xi \to M$ is trivial.

(2) A co-orientable distribution $\xi = \ker \alpha$ is integrable if and only if $\alpha \wedge d\alpha \equiv 0$.

Notice that the contact condition (1.1) is a partial differential relation. The formal version of this relation is

$$\alpha \wedge \omega^n > 0,$$

for some 1-form α and some 2-form ω. This motivates the following definition.

Definition. Let M^{2n+1} be an oriented smooth manifold. An **almost contact structure** on M is a pair (α, ω), defined up to a scalar factor, where α is a nonvanishing 1-form on M, and ω is a non-degenerate 2-form on $\ker \alpha$.

Observe our technique for moving from a partial differential relation to a formal relation: we simply omit any differential conditions. To define the formal homotopy counterpart to a symplectic form, this means dropping the condition that our form be closed.

Speaker: Austin Christian

Definition. An **almost symplectic structure** on a smooth manifold is a non-degenerate 2-form.

Even before considering the formal homotopy versions of contact and symplectic structures, these geometries exhibit a surprising amount of flexibility.

Theorem 1.1 (Gray's stability theorem)**.** *Let ξ_t, $t \in [0,1]$ be a smooth family of contact structures on a manifold M which coincide outside of a compact set. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of M such that $\psi_t^*\xi_t = \xi_0$ for $t \in [0,1]$.*

The upshot of this theorem is that we cannot change the contactomorphism class of our contact structure by a deformation. This contrasts sharply with the situation for Riemannian manifolds, but symplectic structures exhibit the same behavior:

Theorem 1.2 (Moser)**.** *Let ω_t, $t \in [0,1]$ be a smooth family of symplectic forms on a manifold M which coincide outside of a compact set, and suppose that each $\omega_t - \omega_0$ belongs to the same cohomology class with compact support. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of M such that $\psi_t^*\omega_t = \omega_0$ for $t \in [0,1]$.*

We conclude our definitions section by defining an important class of submanifolds of contact or symplectic manifolds.

Definition. Let M be a smooth manifold carrying either a contact structure ξ (if $\dim M = 2n+1$) or a symplectic structure ω (if $\dim M = 2n$). We call a submanifold L **isotropic** if $TL \subset \xi$ in the contact case, or if $\omega|_L \equiv 0$ in the symplectic case. If L is of maximal dimension n, then we call L **Legendrian** in the contact case and **Lagrangian** in the symplectic case.

Some of the earliest h-principles were for immersion or embedding problems; in the contact and symplectic categories, isotropic submanifolds are the relevant objects for these types of problems.

1.2 Gromov's alternative

This section contains some history of the flexible/rigid dichotomy in contact/symplectic topology, as told in [Eli15].

In the late 1960s, Gromov observed several instances of the h-principle in the symplectic and contact categories. For instance, next week we'll discuss Gromov's h-principle for contact structures on open manifolds:

Theorem 1.3 ([Gro69])**.** *Given an open manifold M, a non-vanishing 1-form α_0 on M, and a non-degenerate 2-form ω_0 on $\xi_0 = \ker\alpha_0$, there exists a family of non-vanishing 1-forms α_t on M and a family of non-degenerate 2-forms ω_t on $\xi_t = \ker\alpha_t$, $t \in [0,1]$, such that α_1 is a contact form and $\omega_1|_{\xi_1} = d\alpha_1|_{\xi_1}$.*

In the same paper, Gromov also proved h-principles for Lagrangian immersions and *ϵ-Lagrangian embeddings*, which are embeddings which fail to be Lagrangian by an angle of at most ϵ. Altogether, these results could be taken as growing evidence for the field of symplectic/contact topology being flexible, and Gromov came close to proving this statement in a more precise way:

Theorem 1.4 ([Gro86, Section 3.4.4(H)])**.** *Let (M,ω) be a connected symplectic manifold, and let $\mathrm{Diff} M$ be the group of diffeomorphisms of M, with subgroups*

$$\mathrm{Symp} M \subset \mathrm{Vol} M \subset \mathrm{Diff} M$$

of symplectomorphisms and volume-preserving diffeomorphisms, respectively. Then $\mathrm{Symp} M$ is either C^0-closed in $\mathrm{Diff} M$ or its C^0-closure is $\mathrm{Vol} M$.

This is known as *Gromov's alternative*, and if the latter statement were true, then we would say that Gromov's alternative has reached a flexible resolution. Namely, every volume-preserving diffeomorphism of (M,ω) could be C^0-approximated by symplectomorphisms, and thus problems of symplectic geometry would be reduced to the more flexible problems of volume-preserving geometry. For better or worse, a rigid resolution was reached.

Theorem 1.5 ([Eli81]). *For a connected symplectic manifold (M, ω), the group* $\mathrm{Symp} M$ *is C^0-closed in* $\mathrm{Diff} M$.

Remark. The dates here are a little confusing. It seems that Gromov's alternative was known well before Eliashberg's rigidity result, but the standard citation for Gromov's alternative is his book [Gro86], and Gromov's book cites Eliashberg's announcement [Eli81] of his rigidity result. The contact analogues of Theorems 1.4 and 1.5 are also true (see [MS14]).

The most revolutionary manifestation of rigidity in symplectic topology was the introduction [Gro85] by Gromov of pseudoholomorphic curves. These have made possible possible the construction of Floer theories, Hofer's metric on Hamiltonian diffeomorphisms, Gromov-Witten invariants, and indeed much of the symplectic/contact topology that has been developed over the last 35 years. The stunning success of pseudoholomorphic curve techniques is probably what led Eliashberg to his "holomorphic curves or nothing" philosophy — if an h-principle cannot be disproved via holomorphic curve techniques, then it's probably true.

1.3 Flexibility makes a comeback

As successful as pseudoholomorphic curve techniques have been, there do, in fact, remain flexible problems in symplectic and contact topology. We will pursue this side of the dichotomy by studying the classification of overtwisted contact structures in all dimensions.

1.3.1 Overtwistedness in dimension three

Perhaps the most natural problem in contact geometry is that of classifying contact structures (say, up to isotopy) on an odd-dimensional manifold M. Generally, this problem has proven to be remarkably subtle, but in dimension three there are several things we can say. Martinet [Mar71] showed that every 3-manifold admits a contact structure, and Lutz [Lut71] showed that, in fact, every homotopy class of 2-plane fields on a 3-manifold admits a contact structure.

This is a very natural occasion for the appearance of an h-principle. Thanks to Gray's theorem, a pair ξ, ξ' of contact structures on M are isotopic if and only if they are homotopic as contact structures. The corresponding formal problem asks only if ξ, ξ' are homotopic as 2-plane fields, relaxing the condition of maximal non-integrability. That is:

Must two contact structures on M which are homotopic as 2-plane fields be homotopic as contact structures?

If the answer were yes, we would have an h-principle for contact structures on 3-manifolds, and the classification of contact structures up to isotopy would, in dimension three, be equivalent to the classification of 2-plane fields up to homotopy. Alas, this is not the case.

Theorem 1.6 ([Ben83]). *There exist contact structures on S^3 which are homotopic as 2-plane fields, but which are not isotopic as contact structures.*

Remark. According to [EH01], there are 3-manifolds whose contact structures are determined up to isotopy by their homotopy type as 2-plane fields, but these are very much the exception rather than the rule.

The feature used by Bennequin to distinguish contact structures on S^3 may seem at first an oddity, but an upcoming talk will attempt to motivate the definition of *overtwisted discs* and discuss why they've been so important.

Definition. An embedded disc D in a contact manifold (M, ξ) is an **overtwisted disc** if (1) its boundary ∂D is a Legendrian curve; (2) the surface and contact framings of ∂D agree; (3) the characteristic foliation D_ξ contains a unique singular point in the interior of D.

Remark. The *surface framing* of ∂D is $TD|_{\partial D}$, while the *contact framing* is $\xi|_{\partial D}$.

Definition. A contact structure ξ on a 3-manifold M is called **overtwisted** if (M, ξ) contains an overtwisted disc. A contact structure which is not overtwisted is called tight.

Remark. Observe that overtwistedness is a contactomorphism invariant (and hence an isotopy invariant).

Overtwistedness seems at first a strange property, but it's the property used by Bennequin to distinguish between[1] $(\mathbb{R}^3_{x,y,z}, \ker(dz + xdy))$ and $(\mathbb{R}^3_{r,\theta,z}, \ker(\cos rdz + r\sin rd\theta))$. Perhaps more importantly, overtwisted contact structures obey an h-principle.

Theorem 1.7 ([Eli89])**.** *Two overtwisted contact structures on a 3-manifold which are homotopic as plane fields are homotopic as contact structures.*

In the last decade, several further contact-topological discoveries have been made which have a flexible flavor. A few keywords include *loose Legendrians*, a type of Legendrian embedding which satisfies an h-principle; *flexible Weinstein manifolds*, which are symplectic manifolds built from handles which are attached along loose Legendrians; and *plastikstufe*, a type of embedded object which can play a role similar to that of an overtwisted disc. Some or all of these keywords may make further appearances this quarter, but our first concern will be with the classification of overtwisted contact structures in all dimensions.

1.3.2 Overtwistedness in all dimensions

The purpose of at least the first half of our seminar will be to understand the main results of [BEM15], the first of which is an existence h-principle for contact manifolds in any dimension.

Theorem 1.8 ([BEM15])**.** *Let M be a $(2n+1)$-manifold, $A \subset M$ be a closed set, and ξ be an almost contact structure on M. If ξ is genuine on an open neighborhood of A, then ξ is homotopic relative to A to a genuine contact structure. In particular, any almost contact structure on a closed manifold is homotopic to a genuine contact structure.*

Notice that this generalizes Gromov's h-principle for contact structures on open manifolds of any odd dimension, as well as generalizing work of Martinet and Lutz on closed 3-manifolds.

Next we want to state a somewhat general result from [BEM15], of which the classification of overtwisted contact structures is a corollary. We should first say what it means for a contact manifold (M^{2n+1}, ξ) to be overtwisted. Let us say that a contact manifold is overtwisted if it admits a contact embedding of a piecewise smooth disc D^{2n}_{ot} with a model contact germ ζ_{ot} which we will specify later. In the $n = 1$ case, this model agrees with the previous notion of overtwistedness.

With this working definition of overtwistedness, we can begin thinking about an h-principle. Consider a $(2n+1)$-manifold M, a closed subset $A \subset M$, and an almost contact structure ξ_0 on M which is a genuine contact structure on $\mathcal{O}\mathrm{p}\, A$, some unspecified open neighborhood of A. We write $\mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0)$ for the space of contact structures on M which (1) are overtwisted on $M \setminus A$ and (2) agree with ξ_0 on $\mathcal{O}\mathrm{p}\, A$. By $\underline{\mathrm{Cont}}(M; A, \xi_0)$ we mean the space of almost contact structures which agree with ξ_0 on $\mathcal{O}\mathrm{p}\, A$, relaxing the contact and overtwistedness conditions. Notice that we have an inclusion

$$j\colon\ \mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0) \to \underline{\mathrm{Cont}}(M; A, \xi_0)$$

of our space of contact structures into its formal analogue.

Theorem 1.9 ([BEM15])**.** *The inclusion $\mathrm{Cont}_{\mathrm{ot}}(M; A, \xi_0) \to \underline{\mathrm{Cont}}(M; A, \xi_0)$ is a weak homotopy equivalence.*

Notice that by taking $A = \emptyset$ we have an isomorphism between homotopy classes of overtwisted contact structures on M and homotopy classes of hyperplane fields on M. Applying Gray's stability theorem, we have the following corollary.

Corollary 1.10 ([BEM15])**.** *On any closed manifold M any almost contact structure is homotopic to an overtwisted contact structure which is unique up to isotopy.*

[1] We stated Bennequin's distinction for contact structures on S^3; by removing a point from S^3 we obtain the contact structures on $\mathbb{R}^3$ listed here.

In the time since Bennequin first observed the overtwisted phenomenon in dimension 3, there have been various attempts at generalizing this notion to higher dimensions. Corollary 1.10 is perhaps the most compelling reason for thinking of the generalization offered by Borman-Eliashberg-Murphy as the correct one. Once we have discussed the proof of Theorems 1.8 and 1.9, we will investigate other exotic phenomena which indicate overtwistedness, including *plastikstufe* [Nie06] and perhaps *bLobs* [MNW13] and *overtwisted oranges* [HH18]. The existence of any of these embedded objects in a contact manifold is now known to be equivalent to overtwistedness, and each of them has their own use in establishing properties of overtwisted contact structures.

For instance, plastikstufe are useful because it is known that contact manifolds admitting embedded plastikstufe are not symplectically fillable ([Nie06]) and satisfy the Weinstein conjecture ([AH09]). Both of these properties hold for overtwisted contact manifolds in dimension 3, and because plastikstufe detect overtwistedness, are now known to hold for overtwisted contact manifolds in any dimension.

1.4 Plan for the quarter

As we've said, our first goal this quarter is to understand the proofs of Theorems 1.8 and 1.9. Because these results build on Gromov's h-principle for contact structures on open manifolds ([Gro69]), we will start there. Once we understand Gromov's proof, we will review the overtwisted classification in dimension 3 ([Eli89]) before spending three or so talks sketching the proofs of the main results of [BEM15]. Once we feel comfortable with the classification of overtwisted structures, our goal will be to understand various geometric criteria which are equivalent. We will learn about the relationship between loose Legendrians and overtwisted discs ([CMP19], [Hua17]) before turning to plastikstufe ([Nie06]), bLobs ([MNW13]), and overtwisted oranges ([HH18]). We also plan to use these other criteria to prove important properties about overtwisted contact structures.

2 Gromov's *h*-principle for contact structures on open manifolds

The goal of this talk is to prove the following theorem of Gromov:

Theorem 2.1 ([Gro69]). *Given an open manifold M, let $\operatorname{Cont} M$ denote the space of cooriented contact structures on M, and let $\underline{\operatorname{Cont}}\, M$ denote the space of cooriented almost contact structures. Then the inclusion*

$$\operatorname{Cont} M \hookrightarrow \underline{\operatorname{Cont}}\, M$$

is a weak homotopy equivalence.

This should be compared to Theorem 1.3, both results being referred to as *Gromov's h-principle for contact structures on open manifolds*.

Once we have set up the appropriate language, this result will follow quickly from the *holonomy approximation theorem*, and this talk will essentially be divided into two parts: first we establish enough background to state the holonomy approximation theorem, and then we discuss how to interpret Gromov's result as a holonomic approximation problem.

2.1 Jet bundles

Recall that h-principles are formulated in terms of some partial differential relation. We replace our partial differential relation with an algebraic problem which is formally identical, but in which derivatives are replaced by independent higher-order information. This strategy is made more formal with the introduction of *jet spaces*, the appropriate setting for studying partial differential relations.

Remark. Throughout the rest of this talk (and indeed, much of the rest of this quarter) we will use the notation $\mathcal{O}\mathrm{p}\, A$ to denote an unspecified open neighborhood of some subset $A \subset M$.

Definition. Let $X \to M$ be a fiber bundle over a smooth manifold M. We define the r**-jet bundle** $J^r(X) \to M$ (also denoted $X^{(r)} \to M$) by letting the fiber over $p \in M$ be given by

$$\Gamma(X \to \mathcal{O}\mathrm{p}\, p)/\sim,$$

where we have $s_1 \sim s_2$ for some sections $s_1 \colon U \to X, s_2 \colon V \to X$ if there are local coordinates on $\mathcal{O}\mathrm{p}\, p$ in which s_1 and s_2 have the same Taylor polynomials at p, up to order r.

Where a section of the original bundle $X \to M$ records some piece of information over each point $p \in M$, a section of $J^r(X) \to M$ records that same information, along with higher-order (derivative) information. Notice that each section of $J^r(X) \to M$ induces a section of $X \to M$.

Example 2.2. The notion of jet bundles is best understood through some simple examples.

(1) Given a fiber bundle $X \to M$, consider $J^0(X) \to M$. The fiber over a point $p \in M$ consists of sections of $X \to \mathcal{O}\mathrm{p}\, p$, where we identify $s_1, s_2 \colon \mathcal{O}\mathrm{p}\, p \to X$ whenever $s_1(p) = s_2(p)$. So the fiber is simply X_p, and we see that $J^0(X) = X$.

(2) Consider the 1-jet bundle $J^1(M, \mathbb{R})$ of the trivial bundle $M \times \mathbb{R} \to M$. The fiber over $p \in M$ is given by all sections of $M \times \mathbb{R} \to M$ over $\mathcal{O}\mathrm{p}\, p$, subject to an equivalence relation which identifies sections $f, g \colon \mathcal{O}\mathrm{p}\, p \to \mathbb{R}$ whenever $f(p) = g(p)$ and $df_p = dg_p$. So the fiber over $p \in M$ is given by $\mathbb{R} \times T_p^* M$, and we see that $J^1(M, \mathbb{R}) = \mathbb{R} \times T^* M$.

(3) Similarly, we denote by $J^r(\mathbb{R}^n, \mathbb{R}^m)$ the r-jet bundle of the trivial bundle $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A section of the trivial bundle is simply a map $f \colon \mathbb{R}^n \to \mathbb{R}^m$, and all notions of closeness between such sections are determined pointwise. We may construct from f its *r-jet* at x:

$$J_f^r(x) = (f(x), f'(x), \ldots, f^{(r)}(x)),$$

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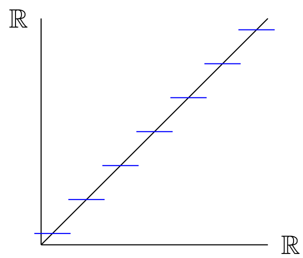


Figure 1: A non-holonomic section of $J^1(\mathbb{R}, \mathbb{R})$.

recording the value of f as well as the values of its various derivatives up to order r at the point x. The assignment $x \mapsto (x, J^r_f(x))$ is then a section of $J^{(r)}(\mathbb{R}^n \times \mathbb{R}^m) \to \mathbb{R}^n$. We call this the r-*jet* of f. Notice that for smooth maps $f, g\colon \mathbb{R}^n \to \mathbb{R}^m$, the r-jets J^r_f and J^r_g agree at a point $x \in \mathbb{R}^n$ if the original sections and all of their derivatives up to order r agree at x. At the same time, there are sections of the r-jet bundle which do not arise from sections of the original bundle.

We can justify the last claim of the above example with a standard counterexample. The 1-jet bundle $J^1(\mathbb{R}, \mathbb{R})$ is given by $\mathbb{R} \times T^*\mathbb{R} \to \mathbb{R}$. If $f\colon \mathbb{R} \to \mathbb{R}$ is any nonzero smooth map, we may construct a section

$$x \mapsto (x, f(x), 0).$$

Because $df \neq 0$, this section does not arise from a section of the original bundle.

Definition. A section F of $J^r(X) \to M$ is said to be **holonomic** if F is the r-jet of the section $M \to X$ which it induces. We call sections which are not holonomic **non-holonomic** or **formal**.

Because they record differential information, jet bundles are a natural setting for the study of differential relations. We think of sections of the jet bundle as (formal) solutions to our differential relation, with genuine solutions being represented by holonomic sections. We may then obtain an h-principle for a problem specified by a differential relation by (1) constructing a jet bundle whose sections give formal solutions to our relation, and (2) perturbing non-holonomic sections to holonomic sections. The latter step can be addressed in some generality, so we treat it first.

2.2 Holonomic approximation

Let us first show that there are formal sections which cannot be well-approximated by holonomic sections. As noted above, we can construct a section $x \mapsto (x, f(x), 0)$ of $J^1(\mathbb{R}, \mathbb{R})$ for any smooth map $f\colon \mathbb{R} \to \mathbb{R}$. Consider the function $f(x) = x$, which generates the section $x \mapsto (x, x, 0)$ depicted in Figure 1. To find a holonomic section which is ϵ-close to this formal section means finding $g\colon \mathbb{R} \to \mathbb{R}$ so that

$$|f(x) - g(x)| < \epsilon \quad \text{and} \quad |0 - g'(x)| < \epsilon,$$

for all $x \in \mathbb{R}$. For $\epsilon < 1$, no such g exists (by, say, the mean value theorem). So we cannot approximate an arbitrary non-holonomic section by holonomic sections.

Perhaps we wonder whether giving ourselves an extra dimension will help. Namely, consider $J^1(\mathbb{R}^2, \mathbb{R})$, and the section defined by

$$(x, y) \mapsto (x, y, x, 0, 0).$$

This section has pointwise information given by $f(x, y) = x$, and tangential information given by 0. Notice that, for the same reason as before, this section cannot be approximated by holonomic sections — or even by sections which are only required to be holonomic when restricted to a neighborhood of the x-axis in $\mathbb{R}^2$. So holonomic approximation seems somewhat hopeless; we cannot guarantee an approximation of a formal section by holonomic sections, even in a neighborhood of a codimension 1 polyhedron. But it turns out that we win if we're willing to perturb the polyhedron.

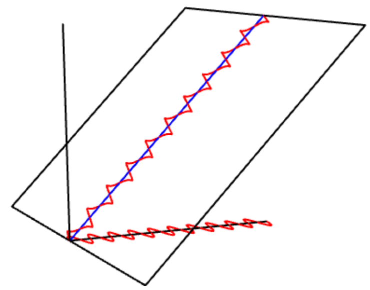


Figure 2: In blue, the pointwise data of a non-holonomic section (with differential data given by 0) over the x-axis. In red, a holonomic section which approximates the non-holonomic section over a polyhedron which is close to the x-axis. (The red section is not contained in the plane.)

Theorem 2.3 (Holonomic approximation theorem)**.** *Consider a fiber bundle $X \to M$ with some fixed metric and let $K \subset M$ be a polyhedron of positive codimension. For any section $F\colon \mathcal{O}\mathrm{p}\, K \to J^r(X)$ and any small $\epsilon, \delta > 0$, there exist a δ-small diffeotopy $h^\tau : M \to M$, $\tau \in [0,1]$ and a holonomic section $\tilde{F}\colon \mathcal{O}\mathrm{p}(h^1(K)) \to J^r(X)$ such that $|\tilde{F}(p) - F(p)| < \epsilon$ for all $p \in \mathcal{O}\mathrm{p}(h^1(K))$.*

Notice what this means for our badly-behaved section $(x, y) \mapsto (x, y, x, 0, 0)$: though we cannot produce a holonomic approximation near the x-axis, we can produce a holonomic approximation near a polyhedron which is C^0-close to the x-axis. Namely, we may perform a large number of small "switchbacks" near the plane $z = x$, approximating our formal section while staying nearly horizontal. See Figure 2.

A proof of the holonomic approximation theorem can be found in [EM02, Chapter 3].

2.3 Proof of Theorem 2.1

Our goal now is to construct a jet bundle whose sections are almost contact structures. We will then use the holonomic approximation theorem to prove Theorem 2.1.

Consider the bundle $\Lambda^1(M) \to M$, whose sections are 1-forms on M. A holonomic section of $J^1(\Lambda^1(M))$ is given by $(\alpha, d\alpha)$, for some $\alpha \in \Omega^1(M)$, and indeed an arbitrary section of $J^1(\Lambda^1(M))$ may be written as (α, ω), with α a 1-form on M and ω a 2-form. Recall that an **almost contact structure** on M is a pair (α, ω), defined up to a scalar factor, with α non-vanishing and ω non-degenerate on $\ker \alpha$. In particular *an almost contact structure leads to a section of $J^1(\Lambda^1(M))$.*

Remark. Notice that an almost contact structure does not lead to a *unique* section of $J^1(\Lambda^1(M))$. But the space of sections representing a fixed almost contact structure is contractible.

Thinking[2] of a point in $J^1(\Lambda^1(M))$ as (α_p, ω_p) for some 1-form α and 2-form ω, we will enforce the non-degeneracy requirement of almost contact structures by considering the subspace

$$\mathcal{R}_{contact} := \{(\alpha_p, \omega_p) \in J^1(\Lambda^1(M)) | \alpha_p \wedge \omega_p^n > 0\},$$

where $\dim M = 2n + 1$.

At last, we are prepared to show that the map

$$\pi_0(\mathrm{Cont}\, M) \to \pi_0(\underline{\mathrm{Cont}}\, M)$$

induced by the inclusion $\mathrm{Cont}\, M \hookrightarrow \underline{\mathrm{Cont}}\, M$ is an isomorphism. We will prove that this induced map is a surjection.

[2] We haven't explained why we're allowed to think of the points of $J^1(\Lambda^1(M))$ in this way. Basically a point in $J^1(\Lambda^1(M))$ should consist of pointwise information — a linear functional — and first-order information — a local variation, which can be represented by a matrix, and then symmetrized. See [EM02, Chapter 10].

We start by making crucial use of our assumption that M is an open manifold. Fix a triangulation of M and a collection of disjoint paths connecting the barycenters of the top-dimensional simplices of our triangulation to ∞. We may use these paths to isotope M away from the barycenters of the top-dimensional simplices, then deformation retract the punctured top-dimensional simplices onto neighborhoods of their boundaries. In this way, M admits a deformation retraction onto a subcomplex consisting of those codimension 1 simplices which do not intersect the paths we've chosen. In particular, we have the following fact.

Proposition 2.4. *Every open manifold M admits a polyhedron $K \subset M$ of positive codimension, an arbitrarily small neighborhood U of K, and an isotopy $\varphi_t \colon V \to V$, $t \in [0,1]$, such that $\varphi_t(M) \subset U$.*

Now think of an almost contact structure (α, ω) as a section of $J^1(\Lambda^1(M))$ and apply the holonomic approximation theorem to this section, with the polyhedron K identified above. The result is a diffeotopy $h^\tau \colon M \to M$, $\tau \in [0,1]$ and a holonomic section $(\tilde{\alpha}, d\tilde{\alpha})$ of $J^1(\Lambda^1(M))$ on $\mathcal{O}\mathrm{p}(h^1(K))$ which is arbitrarily close to $(\alpha,\omega)|_{\mathcal{O}\mathrm{p}(h^1(K))}$. Because we have C^0-wiggle room, we may assume that $(\tilde{\alpha}, d\tilde{\alpha})$ lies in the open set $\mathcal{R}_{contact}$. If ϵ is sufficiently small (and hence $(\tilde{\alpha}, d\tilde{\alpha})$ is sufficiently near (α,ω)), then we may further assume that (α,ω) and $(\tilde{\alpha}, d\tilde{\alpha})$ are homotopic in $\mathcal{R}_{contact}$ (for instance via a linear interpolation).

At last, we extend our holonomic approximation to all of $J^1(\Lambda^1(M))$. With φ_t the isotopy identified by the above proposition, notice that $(\varphi_1^*\tilde{\alpha}, d\varphi_1^*\tilde{\alpha})$ is a holonomic section of $J^1(\Lambda^1(M))$ contained in $\mathcal{R}_{contact}$. We obtain a homotopy from (α,ω) to this holonomic section by concatenating our homotopy from (α,ω) to $(\tilde{\alpha}, d\tilde{\alpha})$ with the homotopy $(\varphi_t^*\tilde{\alpha}, d\varphi_t^*\tilde{\alpha})$. So (α,ω) admits a homotopy to a holonomic section entirely contained in $\mathcal{R}_{contact}$. That is, any almost contact structure is isotopic through contact structures to a genuine contact structure.

So the inclusion $\mathrm{Cont}\, M \hookrightarrow \underline{\mathrm{Cont}}\, M$ induces a surjection on path components; this map is also an injection, and the proof is similar. With a parametric version of the holonomic approximation theorem, one can in fact show that this map is a homotopy equivalence.

3 The overtwisted classification in dimension 3

The goal of today's talk is to understand a proof of the following result.

Theorem 3.1 ([Eli89]). *Let M be a closed 3-manifold. In every homotopy class of 2-plane fields on M, there is a unique overtwisted contact structure.*

Eliashberg proves this as a consequence of theorem which takes slightly more effort to state. Let M be a closed, connected, oriented 3-manifold, and fix an embedded 2-disc $\Delta \subset M$. We fix a contact germ ζ on Δ which makes (Δ, ζ) an *overtwisted disc*. That is, ζ is a 2-plane field defined on Δ which satisfies $\zeta = T\Delta$ at precisely one interior point of Δ — we call this point the **center** of Δ — and $T(\partial\Delta) \subset \zeta$. We now consider two sets of 2-plane fields on M:

- $\mathrm{Cont}_{\mathrm{ot}}(M, \Delta)$ is the set of contact structures on M which agree with ζ on Δ (these are necessarily overtwisted);
- $\mathrm{Dist}(M, \Delta)$ is the set of 2-plane fields which agree with ζ at the center of Δ.

Notice that $\mathrm{Cont}_{\mathrm{ot}}(M, \Delta)$ is a subset of $\mathrm{Dist}(M, \Delta)$.

Theorem 3.2 ([Eli89]). *The inclusion $\mathrm{Cont}_{\mathrm{ot}}(M, \Delta) \hookrightarrow \mathrm{Dist}(M, \Delta)$ induces an injection on path components.*

By work of Lutz and Martinet, this inclusion was already known to induce a surjection on path components; Theorem 3.2 allows us to conclude that the inclusion is a weak homotopy equivalence. In today's talk we will sketch a proof of Theorem 3.2, omitting for brevity's sake the final step of deriving Theorem 3.1. See, for instance, [Gei08, Section 4.7.1] for this last step.

3.1 Outline of the argument

We'll have a proof of Theorem 3.2 if, whenever $\xi_t \in \mathrm{Dist}(M, \Delta)$, $t \in [0, 1]$, is a path connecting $\xi_0, \xi_1 \in \mathrm{Cont}_{\mathrm{ot}}(M, \Delta)$, we're able to construct a path in $\mathrm{Cont}_{\mathrm{ot}}(M, \Delta)$ connecting these two contact structures. Such a path is constructed in three steps.

Step 1. First, we may identify a finite collection $B_0, B_1, \ldots, B_m \subset M$ of disjoint embedded balls away from which we can win our game. That is, we will homotope (rel. endpoints) the family ξ_t so that each ξ_t will satisfy the contact condition outside $B_0, B_1, \ldots, B_m$. These balls will be chosen so that their complement is a neighborhood of the 2-skeleton of a simplicial decomposition of M — reminiscent of Gromov's h-principle — and we will control the restrictions $\xi_t|_{\partial B_i}$ as best we can.

Step 2. Next, the balls $B_0, B_1, \ldots, B_m$ may be connected to obtain a single ball B_t, which varies with t. Again, we carefully control the restriction $\xi_t|_{\partial B_t}$.

Step 3. Finally, we "fill the hole." Because we have controlled the behavior of ξ_t along ∂B_t, we will be able to extend $\xi_t|_{M\setminus B_t}$ to a continuous family ξ_t keeping ξ_0, ξ_1 fixed, and completing our proof.

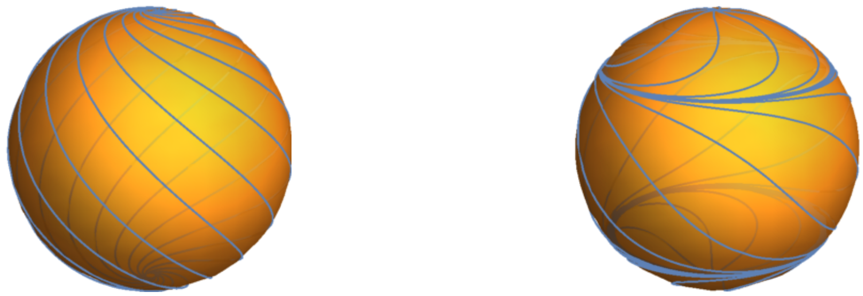
A similar road map will be followed in the higher-dimensional classification, where we will create and fill "universal holes." The real goal of today is to understand how we *would* prove this result, discussing enough basic contact geometry notions that the proof sketch will become accessible. Details will be added to the outline of the argument as time permits.

3.2 Background

3.2.1 Characteristic foliations

We start by explaining how we record the data of a contact structure on a surface Σ in a contact 3-manifold (M^3, ξ). Note that, because ξ is maximally non-integrable, there are no open subsets of Σ on which Σ is

Speaker: Zachary Smith



(a) $S^2 \subset (\mathbb{R}^3, \xi_{\text{tight}})$ (b) $S^2 \subset (\mathbb{R}^3, \xi_{\text{ot}})$

Figure 3: The characteristic foliation of S^2 in a pair of contact structures on $\mathbb{R}^3$.

tangent to ξ. That is, the **singular points** of Σ — the points $p \in \Sigma$ where $T_p\Sigma = \xi_p$ — are isolated. Away from these points, $T\Sigma$ and ξ are transverse 2-plane distributions on Σ, and thus determine a line bundle $\Sigma_\xi := T\Sigma \cap \xi \subset T\Sigma$. This line bundle may be integrated to a 1-dimensional foliation; we call the singular foliation of Σ obtained by including the singular points the **characteristic foliation** of Σ, also denoted Σ_ξ.

Example 3.3. Figure 3 shows the characteristic foliation on S^2 in a pair of contact structures on $\mathbb{R}^3$. In both cases, S^2 has two singular points — one each at the north and south poles. At the north pole, the orientations of ξ and TS^2 agree (for either choice of ξ), while at the south pole they disagree. This means that in both spheres the north pole is a source, while the south pole is a sink. Notice, however, that the foliation in 3b has two closed curves, while the foliation in 3a has none.

We think of a characteristic foliation as the *germ* of the contact structure ξ on the surface Σ. Indeed, the characteristic foliation on Σ determines ξ on some neighborhood $N(\Sigma)$ up to isotopy.

Proposition 3.4. *Let $\Sigma \subset M$ be a surface in a 3-manifold, and let ξ, ξ' be contact structures on M which induce the same characteristic foliation on Σ: $\Sigma_\xi = \Sigma_{\xi'}$. Then there is a neighborhood of Σ on which ξ and ξ' are isotopic.*

The proof of this result is an application of *Moser's trick*.

3.2.2 Overtwisted discs

The essential distinguishing feature between Figures 3a and 3b is that the sphere in Figure 3b contains an *overtwisted disc*.

Definition. An embedded disc Δ in a contact manifold (M, ξ) is an **overtwisted disc** if

(1) its boundary $\partial\Delta$ satisfies $T(\partial\Delta) \subset \xi$;

(2) the surface and contact framings of $\partial\Delta$ agree;

(3) the characteristic foliation Δ_ξ contains a unique singular point in the interior of Δ.

A more concise (if less precise) definition of an overtwisted disc is that it's an embedded disc whose characteristic foliation (and thus its contact germ) matches that in the top region of the sphere in Figure 3b.

We say that a contact manifold (M, ξ) of dimension three is **overtwisted** if it admits an embedded overtwisted disc; otherwise, (M, ξ) is said to be **tight**.

3.2.3 Darboux charts

Much of the reasoning that takes place in the proof of Theorem 3.2 is conducted in a chart which resembles the contact manifold $(\mathbb{R}^3, \ker(dz + x\,dy))$. Remarkably, every point in a contact 3-manifold admits such a chart.

Theorem 3.5 (Darboux)**.** *Let (M, ξ) be a contact manifold of dimension three, and choose $p \in M$. There exist an open neighborhood $U \subset M$ of p and a contactomorphism $\phi\colon (U, \xi|_U) \to (\mathbb{R}^3, \ker(dz + x\,dy))$.*

A proof of this result can be found in any introductory text on contact geometry — e.g., [Gei08].

3.2.4 Foliations on spheres

In our outline we said that we wanted to exert some control over the contact structures ξ_t when restricted to the boundaries ∂B_i of the balls that we choose. What we mean in particular is that we want the characteristic foliation $(\partial B_i)_{\xi_t}$ to be *almost horizontal*.

Definition. Let $S \subset (M, \xi)$ be an embedded sphere, and let S_ξ be its characteristic foliation. We say that S_ξ is **almost horizontal** if

(1) there are exactly two singular points — a *north pole* N, where the orientations of ξ_N and $T_N S$ agree, and a *south pole* S, where orientations disagree;

(2) there are finitely many parallel closed leaves;

(3) all closed leaves are oriented from west to east.

We say that the foliation is **simple** if only the last condition fails.

We can make this last condition slightly more precise by using a transverse arc connecting S to N to define a holonomy map $h\colon [-1,1] \to [-1,1]$. The last condition then requires that this holonomy map be strictly increasing — that is, points move from S towards N.

Example 3.6. Each of the spheres in Figure 3 has a north pole and a south pole, with finitely many closed leaves. Because the sphere in Figure 3a has no closed leaves, the last condition is vacuous. We see that the closed leaves in Figure 3b are oriented east-to-west, so this sphere is not almost horizontal.

We've said that in the proof of Theorem 3.2 we will want to control the foliations $(\partial B_i)_{\xi_t}$, and that we will connect the balls B_i into a single ball B_t. We will make this connection by obtaining a transverse arc[3] γ from the north pole of B_i to the south pole of B_{i+1} and considering the union $B_i \cup N(\gamma) \cup B_{i+1}$, where $N(\gamma)$ is a tubular neighborhood of γ. After smoothing, this union is a ball, with a foliation on its boundary determined up to homeomorphism. If $(\partial B_i)_{\xi_t}$ and $(\partial B_{i+1})_{\xi_t}$ are almost horizontal, then the fact that γ is transverse will ensure that the new foliation is also almost horizontal.

Remark. We choose γ to be a transverse arc so that the characteristic foliation $(\partial N(\gamma))_\xi$ is never vertical, where we are thinking of $\partial N(\gamma)$ as a vertical cylinder. This preserves the property of being almost horizontal.

The reason we take such care with the characteristic foliations $(\partial B_i)_{\xi_t}$ is that the final step of our proof will be to extend a contact structure across the ball B_t. The following lemma (c.f. [Gei08, Lemma 4.7.13]) says that our ability to make this extension will be determined by the foliation $(\partial B_t)_{\xi_t}$.

Lemma 3.7. *Let ξ be a contact structure defined near the boundary ∂B of a 3-ball B, inducing a simple characteristic foliation $(\partial B)_\xi$. Whether or not ξ extends over B as a contact structure is determined by the topological type of $(\partial B)_\xi$.*

We can view this lemma as a consequence of Proposition 3.4, which says that the characteristic foliation of a surface determines the contact structure near that surface.

3.3 A few more details

We can now add some details to the outline sketched above. The details we add follow those found in [Hon, Lecture 5].

Step 1. In this first step, we create the holes away from which we will make ξ_t contact. These holes are constructed to be complementary to a 2-skeleton of M. We first choose B_0 to be a neighborhood of the overtwisted disc Δ, on which the contact structures ξ_t are all assumed to agree. (Because Δ is contractible and the ξ_t agree at the center of Δ, it is straightforward to homotope the ξ_t so that they agree on all of Δ.) Because B_0 is a neighborhood of an overtwisted disc, the characteristic foliation $(\partial B_0)_{\xi_t}$ will be that of Figure 3b.

[3] i.e., γ is positively transverse to ξ

Next, we subdivide M into $n \gg 0$ small cubes C so that, once we identify C with the unit cube in $\mathbb{R}^3$, each Gauss map

$$\phi_{\xi_t} : C \to S^2$$

satisfies $\|\phi_{\xi_t}\| < \frac{\epsilon}{n}$ for some small ϵ. Here ϕ_{ξ_t} is defined by taking an oriented unit normal vector to ξ_t, and $\|\cdot\|$ denotes the supremum norm of the derivative (which is not quote the same as the C^1-norm).

One can check that if $S \subset C$ is a sphere whose principal curvatures are at least $\frac{\epsilon}{n}$, then S_{ξ_t} is almost horizontal, essentially because ξ_t cannot twist quickly enough to create an overtwisted disc. We take $S \subset C$ to be a sphere with almost horizontal characteristic foliation which approximates ∂C. The interior of S will serve as a ball away from which we are contact, so we want to make ξ_t contact on $C \setminus S$. Near the poles of S, we can perturb ξ_t to be contact without difficulty. We now want to make ξ_t contact on a neighborhood $S \times [-1, 1]_s$ which contains ∂C. On this neighborhood we may write ξ_t as the kernel of a 1-form $\alpha = f(s)ds + \beta_s$, where β_s is a 1-form on S (dependent on s). By perturbing β_s, we may make this 1-form satisfy the contact condition. Namely, we have $\alpha \wedge d\alpha > 0$ if

$$f d\beta_s + \beta_s \wedge (df + \frac{d}{ds}\beta_s) > 0.$$

We can then modify β_s to make $\beta_s \wedge (\frac{d}{ds}\beta_s)$ sufficiently large.

One point we've glossed over is that, in order for $S \times [-1, 1]$ to contain ∂C, this thickened sphere must leave C, so we lose our bound on $\|\phi_{\xi_t}\|$. This can be fixed, but we won't fix it here. Finally, there are finitely many cubes C, and thus finitely many balls $B_0, B_1, \ldots, B_n$.

Steps 2 and 3. We can now join the holes we've created. Namely, for each $i = 0, 1, \ldots, n-1$ we may choose a 1-parameter family γ_t^i of transverse arcs connecting the north pole of ∂B_i to the south pole of ∂B_{i+1}, disjoint from $B_0, \ldots, B_n$. We then form a ball B_t by taking the union of $B_0, \ldots, B_n$ with a standard neighborhood $N(\gamma_t^i)$ for each i and smoothing. The characteristic foliation $(\partial B_t)_{\xi_t}$ will be almost horizontal, except for the portion coming from B_0, the neighborhood of the overtwisted disc. Having so standardized $(\partial B_t)_{\xi_t}$, Eliashberg constructs an explicit extension of ξ_t across B_t to complete the proof.

4 Defining overtwistedness in all dimensions

The purpose of this talk is to outline what it means for a contact manifold of arbitrary dimension to be overtwisted in the sense of Borman-Eliashberg-Murphy [BEM15]. As in dimension three, overtwisted contact manifolds are characterized by the existence of an embedded disc with a germ of a contact structure contactomorphic to a model overtwisted disc. Throughout this talk we will slowly build more complicated contact-geometric constructions, culminating in the construction of a model overtwisted disc.

4.1 Standard models

4.1.1 Standard contact structures

Several of our model geometric objects will be constructed in $\mathbb{R}^{2n-1}$ or $\mathbb{R}^{2n+1}$. We want now to identify standard contact structures on these spaces. For $\mathbb{R}^{2n-1}$ we make an identification with $\mathbb{R} \times (\mathbb{R}^2)^{n-1}$ and consider the contact form

$$\lambda_{std}^{2n-1} := dz + \sum_{i=1}^{n-1} r_i^2 dt_i,$$

where (r_i, t_i) are polar coordinates on $\mathbb{R}^2$. We then define $\xi_{std} := \ker \lambda_{std}$.

There are two ways in which we might extend λ_{std}^{2n-1} to $\mathbb{R}^{2n+1}$. Most of the time we will identify $\mathbb{R}^{2n+1}$ with $\mathbb{R}^{2n-1} \times \mathbb{R}^2$ and take

$$\xi_{std} := \ker(\lambda_{std}^{2n-1} + r_n^2 dt_n), \tag{4.1}$$

where (r_n, t_n) are polar coordinates on $\mathbb{R}^2$. We sometimes make the identification $\mathbb{R}^{2n+1} = \mathbb{R}^{2n-1} \times T^*\mathbb{R}$ and use

$$\xi_{std} := \ker(\lambda_{std}^{2n-1} - y_n dx_n), \tag{4.2}$$

where (x_n, y_n) are rectangular coordinates on $\mathbb{R}^2$. We remark that (4.1) and (4.2) are contactomorphic contact structures, but it will prove convenient to choose one over the other for different models.

4.1.2 Star-shaped domains and characteristic foliations

The geometry of our model objects will be controlled by insisting that these objects be transverse to certain vector fields, and by dictating properties of the *characteristic foliations* of the objects, much as in the three-dimensional case.

Definition. We will call a compact domain in $(\mathbb{R}^{2n-1}, \xi_{std})$ **star-shaped** if its boundary is transverse to the vector field $Z = z\partial_z + \sum_{i=1}^{n-1} \frac{1}{2} r_i \partial_{r_i}$. We call a contact closed ball Δ^{2n-1} **star-shaped** if it is contactomorphic to a star-shaped domain in $(\mathbb{R}^{2n-1}, \xi_{std})$.

Remark. Notice that because $\mathcal{L}_Z \lambda_{std} = \lambda_{std}$, flowing along Z preserves ξ_{std}. For this reason, we say that Z is a *contact vector field*.

Recall that in the classification of overtwisted contact structures in dimension three, we concerned ourselves with characteristic foliations on spheres. The higher-dimensional classification will require similar considerations, so we now define characteristic foliations in higher dimensions.

Definition. Let $\Sigma \subset (M, \ker \lambda)$ be a hypersurface in a contact manifold. We call the singular 1-dimensional distribution

$$\ker(d\lambda|_{\ker(\lambda|_\Sigma)})$$

the **characteristic distribution** of Σ, and call the singular foliation to which this distribution integrates the **characteristic foliation** of Σ, denoted Σ_ξ.

Speaker: Eilon Reisin-Tzur

Remark. The characteristic foliation is independent of the contact form λ. Unlike in the 3-dimensional case, the characteristic foliation of Σ alone probably[4] does not determine the contact structure on a neighborhood of Σ. In addition to the characteristic foliation, we need a *transverse contact structure*, which is to say contact structures on hypersurfaces in Σ which are transverse to Σ_ξ.

4.2 Contact shells

As in the three-dimensional case, the overtwisted classification will proceed by creating and filling "universal holes" in our almost contact manifold. As before, we control the germs of the contact structures along the boundaries of these holes, and this section establishes the terminology we will need in order to do this.

The key idea is this: suppose we have an almost contact structure on a $(2n+1)$-manifold M, and that we have made this structure genuinely contact everywhere except on $\mathcal{O}\mathrm{p}(V)$, where $V^{2n-1} \subset M$ is a codimension-2 contact submanifold. We will then want to make the almost contact structure genuine on $\mathcal{O}\mathrm{p}(V)$, without adjusting the structure outside of $\mathcal{O}\mathrm{p}(V)$. In this section we construct general models for extension problems of this type.

4.2.1 Shells and gluing places

Throughout this section, a ball is treated as a domain in some ambient manifold.

Definition. A **contact shell** is an almost contact structure ξ on a ball B which is a genuine contact structure on $\mathcal{O}\mathrm{p}\,\partial B$. We call the contact shell **solid** if ξ is a genuine contact structure on all of B. An **equivalence** between contact shells (B, ξ) and (B', ξ') is a diffeomorphism $g\colon B \to B$ such that

(1) $g_*\xi|_{\mathcal{O}\mathrm{p}\,\partial B'} = \xi'|_{\mathcal{O}\mathrm{p}\,\partial B'}$;

(2) $g_*\xi$ is homotopic rel $\mathcal{O}\mathrm{p}\,\partial B'$ to ξ' through almost contact structures.

Remark. In dimension three, we tracked the data of a characteristic foliation on ∂B, and we remarked that this foliation determines a contact structure on $\mathcal{O}\mathrm{p}\,\partial B$ up to isotopy. Additionally, if ξ is a contact structure defined on $\mathcal{O}\mathrm{p}\,\partial B$, then whether or not ξ can be extended to a contact structure on B is determined by the topological type of the characteristic foliation. (c.f. [Gei08, Lemma 4.7.13]). Contact shells will provide the analogous information in higher dimensions.

Definition. Consider contact shells $\zeta_+ = (B_+, \xi_+)$ and $\zeta_- = (B_-, \zeta_-)$. We say that ζ_+ **dominates** ζ_- if there exist

(1) a shell $\tilde{\zeta} = (B, \xi)$ with an equivalence $g\colon (B, \xi) \to (B_+, \xi_+)$ of contact shells;

(2) an embedding $h\colon B_- \to B$ such that $h^*\xi = \xi_-$ and ξ is genuine on $B \setminus \mathrm{Int}\, h(B_-)$.

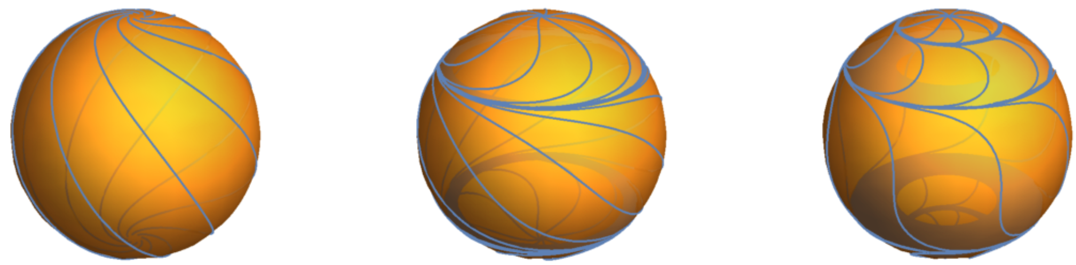
Example 4.1. Certainly if we take concentric balls in a contact manifold, the outer ball is a solid contact shell which dominates the smaller (also solid) contact shell. Figure 4 shows how the characteristic foliation can change via subordination for some balls in $(\mathbb{R}^3, \cos r dz + r \sin r d\theta)$.

Just as we connected contact balls in the overtwisted classification in dimension three, we will want to connect contact shells.

Definition. A (smooth) point $p \in \partial B$ of a contact shell (B, ξ) is called a **gluing place** if $T_p\partial B = \xi_p$. Given gluing places $p_i \in (B_i, \xi_i)$ for $i = 0, 1$, we form the **boundary connected sum** $(B_0 \# B_1, \xi_0 \# \xi_1)$ of (B_0, ξ_0) and (B_1, ξ_1) at p_0, p_1 by making the shells isomorphic in neighborhoods of p_i and performing the usual connected sum.

Remark. During the proof of the overtwisted classification in dimension three, we connected balls in a contact manifold by selecting a transverse arc from the north pole of one ball to the south pole of the other and joining the balls via a tube around this arc. In that setting, the north and south poles were gluing places, and our connecting of the balls corresponded to a boundary connected sum.

[4] I (Austin) don't actually know whether or not anyone's proven this one way or the other.



(a) ∂B^3 (b) $\partial(\frac{3}{2}\pi B^3)$ (c) $\partial(\frac{5}{2}\pi B^3)$

Figure 4: Contact shells in $(\mathbb{R}^3, \xi_{ot})$.

4.2.2 Circular model shells

In dimension three, the final step of the overtwisted classification was to construct a contact structure on a ball with a prescribed contact germ on the boundary. This can be done done by an explicit construction, because the germ on the boundary has been carefully controlled so that it will admit an extension. In this section we define *circular model shells*, in which we will be able to state our extension problems.

The construction begins with a smooth function

$$K\colon \Delta \times S^1 \to \mathbb{R}$$

where $\Delta \subset \mathbb{R}^{2n-1}$ is a compact, star-shaped domain. We assume that $K|_{\partial\Delta\times S^1} > 0$. For any constant $C \in \mathbb{R}$ satisfying $C + \min(K) > 0$, we associate to K a piecewise smooth ball

$$B_{K,C} := \{(x, r, t) \in \Delta \times \mathbb{R}^2 | r^2 \leq K(x, t) + C\} \subset \mathbb{R}^{2n-1} \times \mathbb{R}^2,$$

where (r, t) are treated as polar coordinates on $\mathbb{R}^2$. A good example to keep in mind is $\Delta = [-1, 1] \subset \mathbb{R}$. In this case, $\partial B_{K,C}$ is a (wiggled) cylinder, capped off with a pair of discs $\mathbb{D}^2$. The precise shape (and hence the contact germ) of the cylinder will depend on K. But the contact germ on the discs is independent of K.

We now have a contact germ on $B_{K,C}$; to make $B_{K,C}$ into a contact shell, we want to extend this germ to an almost contact structure. This is most easily done by first introducing some auxiliary functions. We first define $v := r^2$ on $\mathbb{R}^2$, and for each $(x, t) \in \Delta \times S^1$ we have a function

$$\rho_{(x,t)}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$$

satisfying

(a) $\rho_{(x,t)}(0) = 0$, for all $(x, t) \in \Delta \times S^1$;

(b) $\rho_{(x,t)}(v) = v - C$, for $(x, v, t) \in \mathcal{O}p\{v = K(x, t) + C\}$;

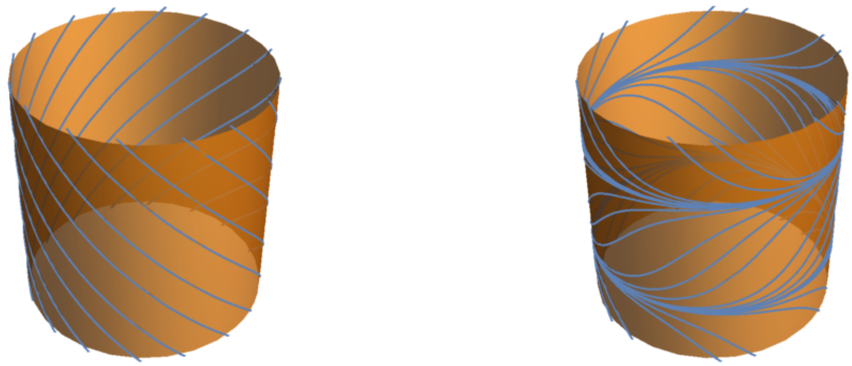
(c) $\partial_v \rho_{(x,t)}(v) > 0$, for $(x, v, t) \in \mathcal{O}p\{v \leq K(x, t) + C, x \in \partial\Delta\}$.

The function $\rho_{(x,t)}$ is meant to measure the rotation of the contact planes as they move from $\{r = 0\} \subset B_{K,C}$ to $\partial B_{K,C}$. Having chosen such a family of functions, we may define an almost contact structure $\eta_{K,\rho} := (\alpha_\rho, \omega)$ by

$$\alpha_\rho := \lambda_{\text{st}} + \rho dt \quad \text{and} \quad \omega := d\lambda_{\text{st}} + dvdt.$$

Notice that

$$\alpha_\rho(\omega)^n = \lambda_{\text{st}}(d\lambda_{\text{st}})^n + n\lambda_{\text{st}}(d\lambda_{\text{st}})^{n-1}dvdt + (d\lambda_{\text{st}})^n dvdt > 0,$$



(a) $K(z,t) = 1$ (b) $K(z,t) = -\cos(\pi z)$

Figure 5: The contact germ on $\{v = K(z,t) + C\}$, with $\Delta = \Delta_{\text{cyl}}$. In each case, the top and bottom of the cylinder is closed with a standard (tight) disc. We have straightened out the surface on the right.

so $\eta_{K,\rho}$ is indeed an almost contact structure. The same calculation shows that $\alpha_\rho(d\alpha_\rho)^n > 0$ whenever $\partial_v \rho > 0$ — that is, the planes must rotate in order for us to have a contact structure. Conditions (b) and (c) ensure that this occurs near $\partial B_{K,C}$, and thus $(B_{K,C}, \eta_{K,\rho})$ is a contact shell.

Lemma 4.2 ([BEM15, Lemma 2.1])**.** *Up to equivalence, the contact shell $(B_{K,C}, \eta_{K,\rho})$ is independent of the choice of ρ and C.*

In light of this fact, we write (B_K, η_K) for the **circle model associated to** the contact Hamiltonian (K, Δ). If $K > 0$, notice that we can let $C = 0$ and $\rho_{(x,t)}(v) = v$ for all $(x,t) \in \Delta \times S^1$. Then $\partial_v \rho > 0$, so (B_K, η_K) is a solid contact ball. Conversely, if K ever vanishes, then $\partial_v \rho$ will vanish, causing the contact condition to fail at some point. So the circle model associated to the contact Hamiltonian (K, Δ) will be a solid contact shell if and only if $K > 0$.

4.2.3 Cylindrical domains

The overtwisted disc will be defined using a particular star-shaped domain which we now define. Let

$$\Delta_{\text{cyl}} := D^{2n-2} \times [-1,1] = \{\Sigma_{i=1}^{n-1} r_i^2 \leq 1, |z| \leq 1\} \subset (\mathbb{R}^{2n-1}, \xi_{st}).$$

For any contact Hamiltonian (K, Δ_{cyl}), the points

$$P_{\pm 1} := (0, \pm 1, 0, t) \in (\partial B_K, \eta_K)$$

in the coordinates $(u, z, r, t) \in \mathbb{R}^{2n-1} \times \mathbb{R}^2$ are gluing places, which we call the **north pole** and **south pole**. A boundary connected sum $(B_K \# B_{K'}, \eta_K \# \eta_{K'})$ is always performed using the north pole of B_K and the south pole of $B_{K'}$.

Example 4.3. Let's consider a contact Hamiltonian (K, Δ_{cyl}), with $n = 1$. Since $\Delta_{\text{cyl}} = [-1,1]$, we have $\lambda_{\text{st}} = dz$, where z is the coordinate on Δ_{cyl}. On $\mathcal{O}\text{p}\{v = K(z,t) + C\}$ we have

$$\alpha_\rho = dz + K(z,t)dt,$$

according to (b). Notice that the characteristic foliation of $\{v = K(z,t) + C\}$ will be horizontal whenever $K(z,t) = 0$. See Figure 5. Notice that Figure 5b has a closed, horizontal orbit for each z-value where $K(z) = 0$, since K does not depend on t.

4.2.4 Ordering contact Hamiltonians

We have a partial order on contact shells given by domination, and we now have a recipe for constructing contact shells from contact Hamiltonians. Let us define a partial order on the contact Hamiltonians by writing

$$(K, \Delta) \leq (K', \Delta')$$

whenever $\Delta \subset \Delta'$,

$$K(x,t) \leq K'(x,t) \text{ for } x \in \Delta, \quad \text{and} \quad 0 < K'(x,t) \text{ for } x \in \Delta' \setminus \Delta.$$

This partial order plays nicely with our construction of circular model shells.

Lemma 4.4 ([BEM15, Lemma 4.1]). *If $(K, \Delta) \leq (K', \Delta')$, then (B_K, η_K) is dominated by $(B_{K'}, \eta_{K'})$.*

4.3 Overtwisted discs

When deciding how to generalize the overtwisted disc to higher dimensions, we must determine which properties of the overtwisted disc in dimension three we want to emulate. The goal of [BEM15] is to generalize Eliashberg's classification result, and thus we define overtwistedness so that it captures the key property which made that result hold. Namely, in the proof of the classification result in dimension three, we had a finite collection $B_1, \ldots, B_n$ of balls, away from which we could make our almost contact structure contact, but we didn't know how to extend the contact structure over these domains. Once these balls were joined with a neighborhood of an overtwisted disc into a single ball, the contact germ on the boundary of this ball could be extended. This is the property we will mimic in higher dimensions. The first step of this mimicry is the following proposition, a proof of which will be sketched in the next talk.

Proposition 4.5 ([BEM15, Proposition 3.1]). *For each dimension $2n+1$, there exists a contact Hamiltonian $(K_{\text{univ}}, \Delta_{\text{cyl}})$ such that the following holds. Let M be any $(2n+1)$-manifold, $A \subset M$ a closed set, and ξ an almost contact structure on M which is genuine on $\mathcal{O}\text{p}\, A \subset M$. Then there exists an almost contact structure ξ' on M, which is homotopic to ξ relative A through almost contact structures, and a finite collection of disjoint balls $B_i \subset M \setminus A$, $i = 1, \ldots, L$, with piecewise smooth boundaries such that ξ' is a genuine contact structure on $M \setminus \cup_{i=1}^{L} \text{Int}(B_i)$ and the contact shells $\xi'|_{B_i}$ are equivalent to $(B_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})$ for $i = 1, \ldots, L$.*

Remark. The function K_{univ} is not uniquely defined by Proposition 4.5. Any contact Hamiltonian for which the conclusions of Proposition 4.5 hold may be called K_{univ}, and Borman-Eliashberg-Murphy do not have a simple criterion for determining whether or not a given function K can be used, except in dimension three. (In dimension three, $K\colon [-1, 1] \to \mathbb{R}$ need only be somewhere negative.)

This proposition performs one half of the classification proof. Namely, this result reduces the problem of homotoping an almost contact structure to a genuine one to the problem of performing this homotopy over the contact shell $(B_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})$. Notice that if (B_K, η_K) is dominated by $(B_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})$, then the homotopy problem can be further reduced to (B_K, η_K), since the almost contact structure is genuine on the annulus between these shells.

Next week we might define what it means for a contact Hamiltonian (K, Δ_{cyl}) to be *special*. The key feature of special contact Hamiltonians is that they produce contact germs which behave much like the overtwisted disc in dimension three. We will make this very vague statement more precise in the next talk, but for now, let us suppose that $K\colon \Delta_{\text{cyl}} \to \mathbb{R}$ is a special contact Hamiltonian and define the contact germ (D_K, η_K). We have a constant $z_D \in (-1, 1)$, specified by the fact that K is special, and define (D_K, η_K) by

$$D_K := \{(x, r, t) \in \partial B_K | z(x) \in [-1, z_D]\} \subset (B_K, \eta_K).$$

For instance, we may take $z_D = 0$ in Figure 5. In all cases, D_K inherits the south pole of ∂B_K. Notice that for Figure 5b, (D_K, η_K) contains an overtwisted disc.

We will explain these ideas more carefully in the next talk, but for now we have the following definition.

Definition. Let K_{univ} be as in Proposition 4.5. An **overtwisted disc** $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}})$ is a $2n$-dimensional disc with a germ of a contact structure such that there is a contactomorphism

$$(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \cong (D_K, \eta_K)$$

where K is some special contact Hamiltonian such that $K < K_{\mathrm{univ}}$. A contact manifold (M^{2n+1}, ξ) is called **overtwisted** if it admits a contact embedding $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \to (M, \xi)$ of some overtwisted disc.

5 Classification of overtwisted structures in all dimensions

The goal of today's talk is to outline the proof of the main theorems of [BEM15]. First, we have the existence of contact structures on manifolds of odd dimension.

Theorem 5.1 ([BEM15, Theorem 1.1]). *Let M be a $(2n+1)$-manifold, $A \subset M$ be a closed set, and ξ be an almost contact structure on M. If ξ is genuine on $\mathcal{O}p\, A \subset M$, then ξ is homotopic relative to A to a genuine contact structure. In particular, any almost contact structure on a closed manifold is homotopic to a genuine contact structure.*

So in fact we see that every homotopy class of almost contact structures on M admits a contact structure. The contact structures produced by Theorem 5.1 are all overtwisted, and the following result says that there is a unique such structure in each homotopy class.

Theorem 5.2 ([BEM15, Theorem 1.2]). *The inclusion $j\colon \mathrm{Cont}_{\mathrm{ot}}(M;A,\xi_0) \to \underline{\mathrm{Cont}}(M;A,\xi_0)$ induces an isomorphism*

$$j_*\colon \pi_0(\mathrm{Cont}_{\mathrm{ot}}(M;A,\xi_0)) \to \pi_0(\underline{\mathrm{Cont}}(M;A,\xi_0))$$

and moreover the map

$$j\colon \mathrm{Cont}_{\mathrm{ot}}(M;A,\xi_0,\phi) \to \underline{\mathrm{Cont}}(M;A,\xi_0,\phi)$$

is a weak homotopy equivalence.

In the above statement, $\phi\colon D_{\mathrm{ot}} \to M \setminus A$ is an embedding of an overtwisted disc, and $\mathrm{Cont}_{\mathrm{ot}}(M;A,\xi_0,\phi)$ and $\underline{\mathrm{Cont}}(M;A,\xi_0,\phi)$ are the subspaces of $\mathrm{Cont}_{\mathrm{ot}}(M;A,\xi_0)$ and $\underline{\mathrm{Cont}}(M;A,\xi_0)$, respectively, for which $\phi\colon (D_{\mathrm{ot}},\zeta_{\mathrm{ot}}) \to (M,\xi)$ is a contact embedding. Combining the above results with Gray's stability theorem, we have the following corollary.

Corollary 5.3 ([BEM15, Corollary 1.3]). *On any closed manifold M any almost contact structure is homotopic to an overtwisted contact structure which is unique up to isotopy.*

Our strategy for proving Theorems 5.1 and 5.2 will mimic the strategy used in dimension three. Namely, Gromov's h-principle for contact structures on open manifolds reduces our existence question to the local problem of extending a contact germ on ∂B^{2n+1} across B^{2n+1}. The first step of the proof is to show that each dimension admits a unique model on which the extension problem must be solved. This corresponds to the step in dimension three where we punch out a finite collection of balls from M, each having almost horizontal characteristic foliations on their boundaries. Next, we connect this universal model to a neighborhood of an overtwisted disc and show that the extension problem can be solved on this new domain.

We will focus on today on proving Theorem 5.1, claiming that the proof of Theorem 5.2 is a parametric version of the same argument. Each step of the proof will use the idea of *conjugating contact Hamiltonians*, so we address this first. We then discuss the two key steps of the proof.

5.1 Conjugation of contact Hamiltonians and overtwisted discs

Recall that in the last talk we defined the *circular model shell* (B_K, η_K) associated to a contact Hamiltonian (K,Δ). In this section we will claim that conjugating a contact Hamiltonian by a contactomorphism of Δ preserves the equivalence class of (B_K,η_K), and use this observation to deduce the following fact.

Proposition 5.4 ([BEM15, Proposition 3.8]). *Every neighborhood of an overtwisted disc in a contact manifold contains a foliation by overtwisted discs.*

Say we have a contact Hamiltonian (K,Δ) and a contactomorphism $\Phi\colon (\Delta,\alpha) \to (\Delta',\alpha')$, where α,α' are contact forms on Δ, Δ'. We want to define a pushforward contact Hamiltonian $(\Phi_* K, \Delta')$. Note that contactomorphisms may change contact forms conformally, so we have a function $c_\Phi\colon \Delta \to \mathbb{R}_{>0}$ such that

$$\Phi^*\alpha' = c_\Phi \alpha.$$

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This motivates the definition

$$(\Phi_* K)(\Phi(x), t) := c_\Phi(x) K(x, t).$$

The contact Hamiltonian $K\colon \Delta \times S^1 \to \mathbb{R}$ is generated by an isotopy ϕ_K^t, and the effect of this pushforward on the isotopy is that of conjugation by Φ. So we refer to $(\Phi_* K, \Delta')$ as the result of **conjugating** K by Φ. Rather importantly, this maneuver preserves equivalence classes of shells.

Lemma 5.5 ([BEM15, Lemma 4.2]). *A contactomorphism $\Phi\colon \Delta \to \Delta'$ between star-shaped domains induces an equivalence $\widehat{\Phi}\colon (B_K, \eta_K) \to (B_{\Phi_* K}, \eta_{\Phi_* K})$ of the contact shells defined by (K, Δ) and $(\Phi_* K, \Delta')$.*

The second ingredient needed to prove Proposition 5.4 is a technical lemma which allows us to scale the z-coordinate of our domain if we're willing to scale the radial directions. The precise statement relies on the definition of special functions. Continuing to blackbox that definition, we define a contactomorphism $C_\delta\colon \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ by scaling the radial components:

$$C_\delta(r_1, \dots, r_{n-1}, t_1, \dots, t_{n-1}, z) = \left(\frac{r_1}{\sqrt{\delta}}, \dots, \frac{r_{n-1}}{\sqrt{\delta}}, t_1, \dots, t_{n-1}, \frac{z}{\delta} \right)$$

whenever $\delta \in \mathcal{O}p\{1\}$. We also define $\Delta_\delta := \{|z| \leq \delta, \Sigma r_i^2 \leq \delta\}$.

Lemma 5.6 ([BEM15, Lemma 4.3]). *Let $K\colon \Delta_{\text{cyl}} \to \mathbb{R}$ be a special contact Hamiltonian and define $K_\delta\colon \Delta_\delta \to \mathbb{R}$ by*

$$K_\delta := K + (\delta - 1).$$

If $\delta < 1$ is sufficiently close to 1, then $\tilde{K}_\delta := (C_\delta)_* K_\delta\colon \Delta_{\text{cyl}} \to \mathbb{R}$ is also a special contact Hamiltonian.

Remark. The proof of Lemma 5.6 is essentially an unwinding of the definition of special contact Hamiltonians.

We are now prepared to prove Proposition 5.4.

Proof of Proposition 5.4. We begin with a fixed neighborhood of an overtwisted disc (D_K, η_K), where (K, Δ_{cyl}) is a special contact Hamiltonian. We may extend this to a neighborhood of $(\partial B_K, \eta_K)$. Now consider $(K_\delta, \Delta_\delta)$, as defined in Lemma 5.6, for $\delta \in [1 - \epsilon, 1]$. For sufficiently small $\epsilon > 0$, we may take the family $\{(\partial B_{K_\delta}, \eta_{K_\delta})\}_{\delta \in [1-\epsilon, 1]}$ is a foliation of our neighborhood of $(\partial B_K, \eta_K)$. We then apply Lemma 5.6 to produce $(\tilde{K}_\delta, \Delta_{\text{cyl}}) \leq (K_{\text{univ}}, \Delta_{\text{univ}})$ by conjugation, and Lemma 5.5 tells us that

$$(\partial B_{K_\delta}, \eta_{K_\delta}) \cong (\partial B_{\tilde{K}_\delta}, \eta_{\tilde{K}_\delta}).$$

So in fact the foliation $\{(D_{K_\delta}, \eta_{K_\delta})\}_{\delta \in [1-\epsilon, 1]}$ of our neighborhood of (D_K, η_K) is a foliation by overtwisted discs. ☐

Remark. The key ingredient to this proof is that, while the contact Hamiltonians K_δ may not satisfy $K_\delta < K_{\text{univ}}$, each $(\partial B_{K_\delta}, \eta_{K_\delta})$ still contains an overtwisted disc because K_δ may be conjugated to a special contact Hamiltonian satisfying $\tilde{K}_\delta < K_{\text{univ}}$. Borman-Eliashberg-Murphy see this as a certain degree of "disorder" among contact Hamiltonians.

5.2 Homotoping to universal holes

In this section we want to comment on the proof of the following proposition, which was stated in the previous talk.

Proposition 5.7 ([BEM15, Proposition 3.1]). *For each dimension $2n + 1$, there exists a contact Hamiltonian $(K_{\text{univ}}, \Delta_{\text{cyl}})$ such that the following holds. Let M be any $(2n + 1)$-manifold, $A \subset M$ a closed set, and ξ an almost contact structure on M which is genuine on $\mathcal{O}p\, A \subset M$. Then there exists an almost contact structure ξ' on M, which is homotopic to ξ relative A through almost contact structures, and a finite collection of disjoint balls $B_i \subset M \setminus A$, $i = 1, \dots, L$, with piecewise smooth boundaries such that ξ' is a genuine contact structure on $M \setminus \cup_{i=1}^L \text{Int}(B_i)$ and the contact shells $\xi'|_{B_i}$ are equivalent to $(B_{K_{\text{univ}}}, \eta_{K_{\text{univ}}})$ for $i = 1, \dots, L$.*

We will sidestep the important details by explaining the easy, 3-dimensional case and claiming that Borman-Eliashberg-Murphy's cleverness takes care of higher dimensions. The key fact in dimension 3 is that circular model shells built from contact Hamiltonians on Δ_{cyl} have a minimal element.

Lemma 5.8 ([BEM15, Lemma 4.7]). *Let (K, Δ) be somewhere negative, with $\Delta = [-1, 1]$. For any other contact Hamiltonian $(\tilde{K}, \Delta)$, there is a contactomorphism $\Phi\colon \Delta \to \Delta$ such that*

$$(\Phi_* K, \Delta) \leq (\tilde{K}, \Delta).$$

In particular, (B_K, η_K) is dominated by $(B_{\tilde{K}}, \eta_{\tilde{K}})$.

Remark. Since $\Delta_{\text{cyl}} = [-1, 1]$, the proof of this fact is a not-terribly-difficult matter of analyzing functions of a single variable (assuming they're time-independent). In higher dimensions the analysis can be much more slippery, and makes identifying a minimal element less likely.

With this in hand, the 3-dimensional version of Proposition 5.7 follows. In particular, Gromov's h-principle for contact structures on open manifolds allows us to assume that our almost contact structure is genuine away from some finite collection of balls. We then apply the above lemma to realize each of these balls as contact shells dominating (B_K, η_K), where $(K, [-1, 1])$ is somewhere negative. But the fact that these balls dominate (B_K, η_K) means that ξ can be made genuine outside some finite number of balls, each of which is equivalent to (B_K, η_K). So $K_{\text{univ}} = K$.

In higher dimensions we are not quite so fortunate, as no minimal contact Hamiltonian up to conjugation has been found. Instead, Borman-Eliashberg-Murphy have the following result.

Proposition 5.9 ([BEM15, Proposition 4.9]). *Let $(K_i, \Delta_{\text{cyl}})$ define contact shells (B_{K_i}, η_{K_i}) for $i = 0, 1$. If there exists $\tilde{\Delta} \subset \text{Int}(\Delta)$ such that*

- *$K_0 \leq K_1$ on $\mathcal{O}p(\Delta \setminus \text{Int}(\tilde{\Delta}))$;*
- *$0 \leq K_1$ on $\mathcal{O}p(\partial\tilde{\Delta})$;*
- *$K_0 \leq 0$ on $\mathcal{O}p(\tilde{\Delta})$;*
- *$K_0|_{\text{Int}(\Delta)} \neq 0$;*

then the contact shell (B_{K_0}, η_{K_0}) is dominated by (B_{K_1}, η_{K_1}).

Essentially, this result says that the only part of (K, Δ) which is relevant to the ordering of contact shells is $K|_{\{K \geq 0\}}$. By analyzing these regions, Borman-Eliashberg-Murphy reduce to a finite collection of *saucers* in each dimension, such that each of the problem areas — where our almost contact structure has not been made contact — can be assumed to match one of these models. The contact Hamiltonian $(K_{\text{univ}}, \Delta_{\text{cyl}})$ is then chosen to be dominated by each of these models, so that Proposition 5.7 is satisfied[5]

5.3 Filling of universal holes

The purpose of this section is to explain what makes special contact Hamiltonians so special, which is the following property. Throughout, the domain Δ is assumed to be Δ_{cyl}.

Proposition 5.10 ([BEM15, Proposition 3.9]). *Let $K \leq K_0$ be two contact Hamiltonians, with K special. Suppose that (B, ξ) is a contact ball such that $(D_K, \eta_K) \subset (\partial B, \xi)$, with the outward coorientation of D_K coinciding with the coorientation of ∂B. Then the contact shell $(B_{K_0} \# B, \eta_{K_0} \# \xi)$ given by performing a boundary connected sum at the north pole of B_{K_0} and the south pole of $D_K \subset \partial B$ is equivalent to a genuine contact structure.*

We will provide a very sketchy argument for why this should be true. First we define, for some $\epsilon > 0$,

$$K' := K - \epsilon \quad \text{and} \quad \Delta' := \{r \leq 1 - \epsilon, |z| \leq 1 - \epsilon\}.$$

[5]This is incredibly sketchy; for more details see Sections 4.3, 6.3, and 8.1 of [BEM15].

Provided $\epsilon > 0$ is small enough that $K'|_{\partial\Delta' \times S^1} > 0$, (K', Δ') is a contact Hamiltonian which is dominated by (K, Δ). In particular, we have an inclusion map

$$(B_{K'}, \eta_{K'}) \hookrightarrow (B_K, \eta_K)$$

which is in fact a subordination map. From this we may define a contact annulus

$$(\mathbf{A}, \xi_{\mathbf{A}}) := (B_K \setminus B_{K'}, \ker \eta_K|_{\mathbf{A}})$$

and thus a contact ball

$$\mathbf{B} := \{(x, r, t) \in \mathbf{A} | z(x) \in [-1, z_D]\},$$

with $\xi_{\mathbf{B}} := \xi_{\mathbf{A}}|_{\mathbf{B}}$. Notice that $(D_K, \eta_K) \subset (\partial \mathbf{B}, \xi_{\mathbf{B}})$. We should think of $(\mathbf{B}, \xi_{\mathbf{B}})$ as a one-sided neighborhood of (D_K, η_K).

Now since $(\mathbf{B}, \xi_{\mathbf{B}})$ is a small, one-sided neighborhood of (D_K, η_K) and $(D_K, \eta_K) \subset (\partial B, \xi)$, we can, by choosing $\epsilon > 0$ sufficiently small in the construction of $(\mathbf{B}, \xi_{\mathbf{B}})$, take $(\mathbf{B}, \xi_{\mathbf{B}})$ to be dominated by (B, ξ). Because $K \leq K_0$, it follows that

$$(B_K \# \mathbf{B}, \eta_K \# \xi_{\mathbf{B}}) \leq (B_{K_0} \# B, \eta_{K_0} \# \xi).$$

It will thus suffice to show that $(B_K \# \mathbf{B}, \eta_K \# \xi_{\mathbf{B}})$ is equivalent to a genuine contact shell. Our ability to show this will rely on the following lemma, the proof of which makes use of Lemma 5.6 and the existence of another type of contactomorphism which may be performed on special contact Hamiltonians. We state this lemma somewhat loosely; for the precise statement, see [BEM15, Lemma 5.4].

Lemma 5.11 (c.f. [BEM15, Lemma 5.4]). *Let (K, Δ) and (K', Δ') be as above, and let $\iota\colon \Delta \hookrightarrow \Delta \# \Delta$ be inclusion into the right hand factor. Then there exists a family of contact embeddings*

$$\Theta_\sigma\colon \Delta \to \mathrm{Int}(\Delta \# \Delta), \quad \sigma \in [0, 1]$$

such that

(*a*) $\Theta_0 = \iota$;

(*b*) $\Theta_\sigma|_{\mathcal{O}\mathrm{p}\{z \in [z_D, 1]\}} = \iota|_{\mathcal{O}\mathrm{p}\{z \in [z_D, 1]\}}$;

(*c*) $\Theta := \Theta_1$ *satisfies* $(\Theta_* K', \Theta(\Delta')) < (K \# K, \Delta \# \Delta)$.

The key idea is that by applying the *transverse scaling* contact embeddings of Lemma 5.6 as well as *twist embeddings*, which we have not described, one can stretch out $\Delta \subset \Delta \# \Delta$ so that it takes up the z-length of the connected sum, and thus accomplish $\Theta_* K' < K \# K$. We will avoid the specifics of this construction, but note that the family of contact embeddings produced by Lemma 5.11 induce

$$\widehat{\Theta}_\sigma\colon (B_{K'}, \eta_{K'}) \to (B_K \# B_K, \eta_{K \# K}),$$

where $\widehat{\Theta} := \widehat{\Theta}_1$ is a subordination map. In particular, $\eta_{K\#K}$ is a genuine contact form away from $\mathrm{Int}(\widehat{\Theta}(B_{K'}))$.

Next, we choose an isotopy $\Psi_\sigma\colon B_K \# B_K \to B_K \# B_K$, $\sigma \in [0, 1]$, such that $\Psi_0 = \mathrm{Id}$, $\Psi_\sigma|_{\mathcal{O}\mathrm{p}(\partial(B_K \# B_K))} = \mathrm{Id}$ for all $\sigma \in [0, 1]$, and which is compatible with $\widehat{\Theta}_\sigma$ in the following sense:

(1) $\Psi_\sigma \circ \widehat{\iota} = \widehat{\Theta}_\sigma$;

(2) $\Psi_\sigma = \mathrm{Id}$ on $\mathcal{O}\mathrm{p}\, \widehat{\iota}\{z \in [z_D, 1]\}$;

(3) $\Psi_1(B_K \# \mathbf{A}) = (B_K \# B_K) \setminus \mathrm{Int}(\widehat{\Theta}(B_{K'}))$.

With this isotopy in hand, we have a family

$$(B_K\#\mathbf{B}, \Psi_\sigma^*(\eta_K\#\eta_K))$$

of equivalent contact shells. Notice that

$$(B_K\#\mathbf{B}, \Psi_0^*(\eta_K\#\eta_K)) = (B_K\#\mathbf{B}, \eta_K\#\eta_{\mathbf{B}}) = (B_K\#\mathbf{B}, \eta_K\#\xi_{\mathbf{B}})$$

is the contact shell of interest to us, and that

$$(B_K\#\mathbf{B}, \Psi_1^*(\eta_K\#\eta_K))$$

is genuine because $\Psi_1(B_K\#\mathbf{B})$ avoids $\mathrm{Int}(\widehat{\Theta}(B_{K'}))$, by (3). This proves Proposition 5.10.

5.4 Proof of Theorem 5.1

At last we can deduce Theorem 5.1 from Propositions 5.4, 5.7, and 5.7.

With $A \subset M$ and ξ as stated in Theorem 5.1, let $B \subset M \setminus A$ be a ball with piecewise smooth boundary. We may homotope ξ to be genuine on B, with $(D_{\mathrm{ot}}, \eta_{\mathrm{ot}}) \subset (\partial B, \xi)$. Note: this is not a homotopy rel ∂B, so we can homotope any almost contact structure on B to any other. We're just pushing our lack of control off of B.

Now ξ is genuine on $A \cup B$, so our perturbations should be fixed on $\mathcal{O}\mathrm{p}(A \cup B)$. We use Proposition 5.7 to produce

$$B_1, \ldots, B_N \subset M \setminus (A \cup B) \quad \text{with} \quad (B_i, \xi|_{B_i}) \cong (B_{K_{\mathrm{univ}}}, \eta_{K_{\mathrm{univ}}})$$

such that ξ can be deformed to be genuine in the complement of $A \cup B \cup B_1 \cup \cdots \cup B_N$.

Next, we choose a ball $B_i' \subset \mathrm{Int}(B)$ which will be paired with B_i, for $i = 1, \ldots, N$. Namely, we choose these B_i' disjoint such that $(D_{\mathrm{ot}}^i, \eta_{\mathrm{ot}}^i) \subset (\partial B_i', \xi)$, and by Proposition 5.4 we can choose them such that we have isomorphisms

$$(B_i\#B_i', \xi|_{B_i\#B_i'}) \cong (B_i\#B_i', \xi|_{B_i}\#\xi|_{B_i'}) \cong (B_{K_{\mathrm{univ}}}\#B_i', \eta_{K_{\mathrm{univ}}}\#\xi|_{B_i'}).$$

Finally, we have $(D_{\mathrm{ot}}^i, \eta_{\mathrm{ot}}^i) = (D_{K_i}, \eta_{K_i})$ for some special contact Hamiltonian $K_i < K_{\mathrm{univ}}$ for each $i = 1, \ldots, N$. Proposition 5.10 then allows us to homotope $\xi|_{B_i\#B_i'}$ to a genuine contact structure on $B_i\#B_i'$, keeping the boundary fixed, and we have homotoped our almost contact structure rel A to a genuine contact structure.

6 Loose Legendrian embeddings

Our primary work so far this quarter has been to establish an *h*-principle for overtwisted contact structures — indeed, we were careful to define overtwistedness so that this *h*-principle will hold. This week we want to discuss an *h*-principle for embeddings of submanifolds into contact manifolds. In particular, we want to discuss *Legendrian embeddings*; recall the following definition.

Definition. A **Legendrian embedding** $f\colon \Lambda \to (M, \xi)$ into a contact manifold of dimension $2n+1$ is a topological embedding such that $\dim \Lambda = n$ and $df_p(T_p\Lambda) \subset \xi_{f(p)}$ for all $p \in \Lambda$.

In dimension three, experts have known for some time of an *h*-principle for a certain class of Legendrian knots. A Legendrian knot $L \subset (M, \xi)$ in a contact 3-manifold is called **loose** if $(M \setminus L, \xi_{M\setminus L})$ is overtwisted. Then we have the following result.

Theorem 6.1 ([Etn10, Theorem 1.4])**.** *Let $f_0, f_1\colon S^1 \to (M^3, \xi)$ be two loose Legendrian embeddings which are isotopic as topological knots and have the same Thurston-Bennequin and rotation numbers. Then $f_0(S^1)$ and $f_1(S^1)$ are Legendrian isotopic knots.*

We haven't defined the Thurston-Bennequin or rotation numbers of a Legendrian knot, but they record the *formal Legendrian isotopy type* of a Legendrian knot.

Definition. Let Λ be a smooth n-manifold, (M, ξ) a contact $(2n+1)$-manifold. A **formal Legendrian embedding** is a pair (f, F_s), where $f\colon \Lambda \to M$ is a smooth embedding and $F_s\colon T\Lambda \to TM$ is a homotopy of bundle maps covering f and satisfying (1) $F_0 = df$; (2) F_s is fiberwise injective for all $s \in [0,1]$; (3) the image of F_1 is contained in ξ and is Lagrangian with respect to the linear conformal symplectic structure on ξ.

In this language, Theorem 6.1 says that if a pair of loose Legendrian embeddings is formally isotopic, then in fact the embeddings are Legendrian isotopic. Our goal today is to understand loose Legendrian embeddings in all dimensions. Namely, we will define a notion of *loose Legendrian* such that the following *h*-principle holds.

Theorem 6.2 ([Mur12, Theorem 1.2])**.** *Suppose $n \geq 2$, and let $f_0, f_1\colon \Lambda \to (M^{2n+1}, \xi)$ be two loose Legendrian embeddings which are formally isotopic. Then they are Legendrian isotopic.*

Notice that in dimension three, loose Legendrians may only appear in overtwisted manifolds. This is not so in higher dimensions, where all contact manifolds contain loose Legendrian embeddings, and indeed loose Legendrians are C^0-dense in any fixed formal Legendrian isotopy class. Nonetheless, the *h*-principles that exist for loose Legendrians and overtwistedness suggest that these notions should be related in some way. We have the following result.

Theorem 6.3 ([CMP19, Theorem 1.1])**.** *Let Λ_0 be the standard Legendrian unknot inside a contact manifold (M, ξ). Then (M, ξ) is overtwisted if and only if Λ_0 is a loose Legendrian.*

This is a consequential result, as checking the definition of overtwistedness given over the last two talks is not really a tractable way to show that a contact manifold is overtwisted; this result gives a simple criterion. Unfortunately its proof requires some background material on flexible Weinstein manifolds which we do not yet have. Today we will focus on sketching a proof of Theorem 6.2.

The strategy of our proof sketch will require an interpretation of formal Legendrian embeddings as sections of the 1-jet space $J^1(\Lambda)$. After introducing this idea, we will explain a result of Eliashberg-Mishachev [EM09] which allows us to approximate families of smooth, codimension 1 embeddings by *wrinkled embeddings*. Applying this result to our sections of $J^1(\Lambda)$ allows us to replace a family of formal Legendrian embeddings with a family of *wrinkled Legendrian embeddings*, which satisfy the Legendrian condition but fail to be smooth. The final step of our proof strategy is to regularize the singularities that have appeared so far; this is the point at which the looseness of our Legendrian embeddings is crucial. We will gloss over the details of this statement, but *loose Legendrians are those which appear as the regularization of wrinkles*.

Speaker: Austin Christian

6.1 Background

6.1.1 Front projections

Recall that for a smooth n-manifold Λ, $J^1(\Lambda)$ denotes the 1-jet bundle of the trivial vector bundle $\mathbb{R} \times \Lambda \to \Lambda$. This space is canonically diffeomorphic to $\mathbb{R} \times T^*\Lambda$, and carries the contact form $dz - \lambda_{\mathrm{std}}$. Here λ_{std} is the tautological 1-form on $T^*\mathbb{R}$; in local coordinates (q_i, p_i) on $T^*\Lambda$, $\lambda_{std} = \Sigma_i p_i dq_i$.

Notice that the natural projection $\pi\colon J^1(\Lambda) \to \mathbb{R} \times \Lambda$ is a *Legendrian submersion*; that is, π is a submersion with the property that $\ker d\pi \subseteq \xi$ at every point of $J^1(\Lambda)$. We call π the *front projection* of $J^1(\Lambda)$. Observe that if $\Lambda' \subseteq J^1(\Lambda)$ is a Legendrian submanifold, then Λ' may be recovered from $\pi(\Lambda')$ using the differential data. For instance, $(\mathbb{R}^{2n+1}_{\mathrm{std}}, dz - \Sigma_i y_i dx_i) = J^1(\mathbb{R}^n)$, and a Legendrian can be recovered from its front projection by setting $y_i = \frac{\partial z}{\partial x_i}$.

6.1.2 Graphical submanifolds

Given a genuine Legendrian embedding $f\colon \Lambda \to (M, \xi)$, standard neighborhood results of contact geometry give us a neighborhood of $f(\Lambda) \subseteq (M, \xi)$ which is contactomorphic to a neighborhood of the zero section of $J^1(\Lambda)$. For a more general smooth embedding f, we do not have such a neighborhood, but the following result says that we can, after an isotopy, identify a neighborhood of a *formal* Legendrian embedding (f, F_s) with an open neighborhood of some section of $J^1(\Lambda)$, using the family F_s to make the identification.

Proposition 6.4. *Let (f, F_s) be a formal Legendrian embedding of Λ into (M, ξ). After a smooth isotopy from f to $\tilde{f}$, we can choose an open set $U \subseteq M$ containing $\tilde{f}(\Lambda)$ and a map $\varphi\colon U \to J^1(\Lambda)$ which is a contactomorphism onto its image, so that $\pi \circ \varphi \circ \tilde{f}$ is the identity map on Λ.*

The upshot is that we are left to analyze formal Legendrian embeddings (f, F_s) as sections $f\colon \Lambda \to J^1(\Lambda)$ of $J^1(\Lambda)$, and in this language we hope to approximate an arbitrary section by holonomic sections.

6.2 Wrinkles

The goal of this section is to define *wrinkled Legendrian embeddings*. These are topological embeddings which are Legendrian wherever they are smooth, but which may have prescribed singularities. We begin by introducing *wrinkled embeddings*, identifying the singularity types allowed for codimension 1 embeddings, and then extend this discussion to Legendrian embeddings.

6.2.1 Wrinkled embeddings

All of the singularity types allowed in wrinkled embeddings are built from the **zig-zag**. This is a plane curve $\psi\colon \mathbb{R} \to \mathbb{R}^2$ which Murphy [Mur12] defines by

$$\psi(u) = (\psi^x(u), \psi^z(u)) = \left(u^3 - u, \frac{9}{4}u^5 - \frac{5}{2}u^3 + \frac{5}{4}u\right),$$

and which is pictured in Figure 6. Notice that the tangent planes of ψ are well-defined, in spite of the singular points $\{3u^2 = 1\}$. One way to express this is by observing that the map

$$Gd\psi\colon \mathbb{R} \to \mathrm{Gr}_{2,1}$$

extends to a smooth function on $\mathbb{R}$. Here $Gd\psi$ sends a point in $\mathbb{R}$ to the image of $d\psi$, which is a 1-plane in $\mathbb{R}^2$. Each of the singularity types that we describe will have this property, and thus so will wrinkled embeddings.

The next singularity type we describe is the **unfurled swallowtail**. First, Murphy defines a family of rescalings $\psi_\delta\colon \mathbb{R} \to \mathbb{R}^2$ for all $\delta \in \mathbb{R}$. When $\delta < 0$, ψ_δ is a smooth curve, so we think of ψ_δ as "pulling ψ tight." We use ψ_δ to define a hypersurface $\mathbb{R}^n \to \mathbb{R}^{n+1}$ according to

$$(x_1, \ldots, x_{n-1}, u) \mapsto (x_1, \ldots, x_{n-1}, \psi_{x_{n-1}}(u)).$$

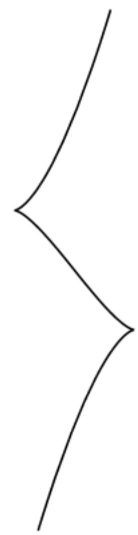


Figure 6: The zig-zag $\psi\colon \mathbb{R} \to \mathbb{R}^2$.

This hypersurface has codimension 1 singular set $\{3u^2 = x_{n-1}\}$, much of which consists of cusp singularities. The unfurled swallowtail is the codimension 2 singularity $\{x_{n-1} = u = 0\}$. For $n = 2$, the unfurled swallowtail can be seen in Figure 7a. Notice that in the foreground, where $x_1 > 0$, a slice of the surface is the zig-zag, while along $x_{n-1} = 0$, a slice is a smooth curve.

Another codimension 2 singularity is given by the **wrinkle**. A wrinkle has a singular set which is a sphere, with cusp singularities on the two hemispheres of this sphere, and unfurled swallowtails along the equator. Namely, we define $w\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ by

$$w(x, u) = (x, \psi_{1-|x|^2}(u))$$

and observe that the singular set is the sphere $\{|x|^2 + 3u^2 = 1\}$. In case $n = 2$, a wrinkle is depicted in Figure 7b, where we can see that the singular set is a circle with unfurled swallowtails at each end and cusps along the hemispheres.

In parametric families of embeddings we allow **embryo** singularities, which are modeled by $w_t(x, u) = (x, \psi_{t-|x|^2}(u))$ at $\{x = u = t = 0\}$. These singularities correspond to the (dis)appearance of wrinkle singularities in our parametric family, and are codimension 1 in time and isolated in space.

Definition. A **wrinkled embedding** is a codimension 1 topological embedding $f\colon V^n \to W^{n+1}$ which is smooth away from a singular set which consists of a finite collection of codimension 1 spheres $S_j^{n-1} \subseteq V$, which bound discs D_j^n. Near S_j^{n-1}, f is modeled on a wrinkle. In k-parametric families, wrinkled embeddings are allowed to have embryo singularities.

Wrinkled embeddings are important because, if we're willing to accept the mild singularities they carry, our holonomic approximation dreams can be fulfilled.

Theorem 6.5 ([EM09],[Mur12, Theorem 3.2]). *Let V^n and W^{n+1} be manifolds, and let $f_t\colon V \to W$ be a parametric family of smooth embeddings, $t \in D^k$. Let $G_t^s\colon V \to \mathrm{Gr}_n(W)$ be a smooth homotopy of maps covering f_t, so that $G_t^0 = Gdf_t$. Then there is a family of wrinkled embeddings $F_t^s\colon V \to W$, so that $F_t^0 = f_0$, and for all $s \in [0, 1]$, F_t^s is C^0-close to f_t, and GdF_t^s is C^0-close to G_t^s.*

(a) An unfurled swallowtail.

(b) A wrinkle.

Figure 7: Codimension 2 singularities.

6.2.2 Wrinkled Legendrian embeddings

It would be nice if we could obtain a result like Theorem 6.5 for formal Legendrian embeddings, so that all formal Legendrians could be well-approximated by genuine Legendrian embeddings. In this subsection we begin identifying the extent to which this is (im)possible.

Let us first consider the $n = 1$ case. Suppose we have a formal Legendrian embedding (f, F_s) into the standard contact space $J^1(\mathbb{R})$, so that $f\colon \Lambda \to J^1(\mathbb{R})$. Then we have a smooth map $\pi \circ f\colon \Lambda^1 \to \mathbb{R}^2_{x,z}$ which we will assume is a smooth embedding, and a homotopy $d\pi \circ F_s\colon \Lambda^1 \to T\mathbb{R}^2_{x,z}$ of maps covering f. Then Theorem 6.5 produces a family of wrinkled embeddings

$$\tilde{F}_s\colon \Lambda \to \mathbb{R}^2_{x,z}$$

with $\tilde{F}_0 = f$, $\tilde{F}_s$ C^0-close to f, and $Gd\tilde{F}_s$ C^0-close to F_s. Now since each $\tilde{F}_s$ is a wrinkled embedding of dimension 1, the only sort of singularity that can appear is a cusp, modeled on $t \mapsto (t^2, t^3) \in \mathbb{R}^2_{x,z}$. Since the cusp lifts to a Legendrian arc in $J^1(\mathbb{R})$ (namely, $(t^2, \frac{3}{2}t, t^3)$), $\tilde{F}_s$ lifts to a Legendrian embedding in $J^1(\mathbb{R})$ which well-approximates f. (To do this carefully, we need to make sure that our wrinkled Legendrian embedding is nowhere vertical.)

More generally, we want to approximate sections $f\colon \Lambda \to J^1(\Lambda)$, for some n-manifold Λ. For $n \geq 2$, the wrinkled embeddings produced by Theorem 6.5 can have unfurled swallowtail and embryo singularities, so we must consider whether these singularities, as maps $\mathbb{R}^n \to \mathbb{R}^{n+1}$, lift to immersions $\mathbb{R}^n \to J^1(\mathbb{R}^n)$. On this front we are not so fortunate: for the unfurled swallowtail, one may compute that the lift satisfies

$$y_i = \begin{cases} 0, & i \leq n-2 \\ 5u(x_{n-1} - u^2)/2, & i = n-1 \\ 15(u^2 - \frac{1}{3}x_{n-1})/4, & i = n \end{cases}.$$

This smooth map fails to be an immersion when $x_{n-1} = u = 0$; in particular, the differential has kernel spanned by ∂_u over this set. This motivates the definition of *wrinkled Legendrian embeddings*, which allows for smooth maps whose front projections are wrinkled.

Definition. Let Λ be a smooth n-manifold and (M, ξ) a contact $(2n+1)$-manifold. A **wrinkled Legendrian embedding** is a smooth map $f\colon \Lambda \to (M, \xi)$ which is a topological embedding and satisfies the following.

- The image of df is contained in ξ everywhere, and df is full rank outside of a subset of codimension 2.
- The codimension 2 singular set is required to be diffeomorphic to a disjoint union of $(n-2)$-spheres $\{S^{n-2}_j\}$, called **Legendrian wrinkles**.
- Each S^{n-2}_j must be contained in a Darboux chart U_j, so that $\Lambda \cap U_j$ is diffeomorphic to $\mathbb{R}^n$, and the front projection

$$\pi_j \circ f\colon \Lambda \cap U_j \to \mathbb{R}^{n+1}$$

of f is a wrinkled embedding, smooth outside of a compact set.

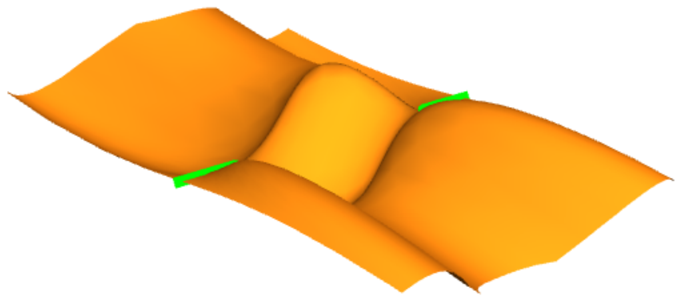


Figure 8: A marking (in green) near a wrinkle.

Parametric families of wrinkled Legendrians are allowed to have **Legendrian embryos**, which are Legendrian lifts from the front projection of embryo singularities.

The Darboux charts U_j are important in that they allow for a *regularization* process. Outside of the Darboux charts, a wrinkled Legendrian is genuinely Legendrian, and within the charts there is an explicit construction, canonical up to homotopy, which allows us to associate a formal Legendrian embedding to any wrinkled Legendrian. Then Theorem 6.5 yields the following result formal Legendrian embeddings.

Theorem 6.6 ([Mur12, Proposition 3.4])**.** *Let $(f_t, F_{s,t})$ be a parametric family of formal Legendrian embeddings $\Lambda \to (M, \xi)$, $t \in D^k$. This family is homotopic through formal Legendrian embeddings to a family $\overline{f}_t \colon \Lambda \to (M, \xi)$ of wrinkled Legendrian embeddings. If $(f_t, F_{s,t})$ is already a wrinkled Legendrian embedding on a closed subset $A \subseteq \Lambda \times D^k$, then we can take $\overline{f}_t = f_t$ on A.*

Remark. It's important to note that the formal isotopy class of a wrinkled Legendrian embedding depends on the Darboux charts U_j which cover the wrinkles, and thus the association of a formal Legendrian embedding to a wrinkled Legendrian may not realize all formal Legendrian isotopy types. In particular, we may not conclude from Theorem 6.6 that every formal Legendrian embedding is isotopic to a genuine Legendrian embedding.

6.3 Loose Legendrians

At this point, brevity demands that we be rather vague. Theorem 6.6 allows us to homotope a family of formal Legendrian embeddings to a family of wrinkled Legendrian embeddings, and next we would like to remove the singularities from this family. There is no canonical way to do this, but Murphy resolves singularities using *markings*.

Definition. Let $f \colon \Lambda \to (M, \xi)$ be a wrinkled Legendrian. A **marking** for f is a compact codimension 1 embedded submanifold $\Phi \subseteq \Lambda$, so that the boundary of Φ is a disjoint union of spheres which are mapped via f to a subset of the Legendrian wrinkles. We further require that in the local models $(\pi_j \circ f)(\Lambda \cap U_j) \cong w(\mathbb{R}^n)$ determined by the wrinkled structure of f, Φ is given as $\{u = 0, |x| > 1\}$, and that the interior of Φ is disjoint from the singular set of f. For parametric families of wrinkled Legendrians, there are additional, technical requirements on Φ which we omit.

According to Murphy, the basic idea is that zig-zags are acceptable in a front projection, but the codimension 2 singularities which occur when these zig-zags are pulled tight are not. Instead of pulling the zig-zags tight, the zig-zags are allowed to persist throughout Λ, and "the marking Φ is just some formal data that tells us where to put the tiny zig-zags in a consistent way." ([Mur12, Section 4.1]) See Figure 8, which depicts a marking near a wrinkle. The wrinkle will be regularized by modifying this surface so that it has tiny zig-zags along the marking, giving us a front projection which lifts to a smooth Legendrian.

Avoiding the details of this singularity resolution, we now claim that loose Legendrians are defined so that they look like resolutions of some wrinkled Legendrian along some marking. Throughout the rest of this section, $\Lambda_0 \subseteq (\mathbb{R}^3, \xi_{\text{std}})$ will denote the Legendrian lift of the zig-zag $\psi(\mathbb{R}) \subset \mathbb{R}^2_{x,z}$. Notice that if $Z \subseteq T^*Q$ is the zero section of the cotangent bundle of some closed manifold Q, then $\lambda_0 \times Z$ is a Legendrian submanifold of $(B^3 \times \mathcal{O}\mathrm{p}(Z), \ker(\alpha_{\text{std}} + \lambda_{\text{std}}))$, since $Z \subseteq \mathcal{O}\mathrm{p}\, Z$ is Lagrangian. Legendrians constructed in this way will represent our model loose Legendrians.

Definition. Let Q be a closed manifold, and let $Z \subseteq T^*Q$ be the zero section of its cotangent bundle. With the contact form $\ker(\alpha_{\text{std}} + \lambda_{\text{std}})$, the pair $(B^3 \times \mathcal{O}\mathrm{p}(Z), \Lambda_0 \times Z)$ is said to be a *loose chart*. We call a Legendrian $\Lambda \subseteq (M, \xi)$ *loose* if $\dim(M) \geq 5$ and there is an open set $V \subseteq M$ such that $(V, V \cap \Lambda)$ is contactomorphic to some loose chart.

An important technical property had by loose Legendrians (essentially because they are resolutions of wrinkles) is the following.

Proposition 6.7 ([Mur12, Proposition 4.4]). *A loose chart contains two disjointly embedded loose charts, and therefore a loose chart contains infinitely many disjointly embedded loose charts.*

6.4 Proof of the *h*-principle

We will derive Theorem 6.2 as a consequence of a more general theorem, a proof of which we are now prepared to sketch. For a fixed loose chart (U, Λ_ℓ) in a contact manifold (M^{2n+1}, ξ) and a fixed, open disc D^n in a smooth manifold Λ, we let $\mathcal{L}^{\text{form}}_\ell(\Lambda, U)$ be the space of all formal Legendrian embeddings with fixed loose chart (U, Λ_ℓ). Namely, $f \colon \Lambda \to M$ satisfies $f^{-1}(U) = D^n$, (f, F_s) is genuine on D^n, and $f(D^n) = \Lambda_\ell$.

Theorem 6.8 ([Mur12, Theorem 1.3]). *Fix $k > 0$ and $n \geq 2$, and for $t \in D^k$ let $(f_t, F_{s,t})$ be a smooth family in $\mathcal{L}^{\text{form}}_\ell(\Lambda, U)$ so that $(f_t, F_{s,t})$ is a genuine Legendrian embedding for all $t \in \partial D^k$. Then the family $(f_t, F_{s,t})$ is isotopic through formal Legendrian embeddings, rel ∂D^k, to a family of genuine Legendrian embeddings.*

Idea of proof. First, by applying Theorem 6.6, we may replace the family $(f_t, F_{s,t})$ with a family of wrinkled Legendrian embeddings. Essentially we are trading in our original, smooth family for a family which fails to smooth, but is Legendrian. By assumption, $(U, f_t(\Lambda))$ is a fixed loose chart, and there is a finite number $K \geq 0$ of codimension 1 submanifolds of D^k where embryo singularities appear. According to Proposition 6.7, we may choose disjoint subsets $U_1, \ldots, U_K \subseteq U$ such that each intersection $f_t(\Lambda) \cap U_i$ is a loose chart. The real work of the proof (which we avoid doing) is then to define a new family of Legendrian embeddings which agrees with f_t away from $\cup_{i=1}^K U_i$, but which replaces each loose chart $f_t(\Lambda) \cap U_i$ with a standard loose chart which admits an explicit smoothing. One then directly verifies that this new family of genuine Legendrian embeddings is isotopic to the original family through formal Legendrian embeddings. □

Proof of Theorem 6.2. If f_0 and f_1 admit a formal Legendrian isotopy which is contained in some $\mathcal{L}^{\text{form}}_\ell(\Lambda, U)$, then we can directly apply the above result to produce our Legendrian isotpoy. But if the charts witnessing looseness of f_0 and f_1 are not the same, then we must produce such a formal Legendrian isotopy. First, choose a contact isotopy φ_t between the loose charts of f_0 and f_1 (possible since these charts are Darboux balls). Then we have an isotopy $\tilde{f}_t := \varphi_t \circ f_t$ and a Darboux ball $U \subseteq (M, \xi)$ such that

$$\tilde{f}_0^{-1}(U) = \tilde{f}_1^{-1}(U) = D^n \subseteq \Lambda, \quad \tilde{f}_0|_{D^n} = \tilde{f}_1|_{D^n},$$

and each of $(U, \tilde{f}_0(D^n))$ and $(U, \tilde{f}_1(D^n))$ is a loose chart. We can then smoothly isotope $\tilde{f}_t$ rel ∂D^1 to a family g_t satisfying $g_t^{-1}(U) = D^n$ and $g_t|_{D^n} = \tilde{f}_0$ for all $t \in D^1$. We now have a smooth isotopy from f_0 to f_1 with fixed loose chart. Finally, we realize g_t as a formal Legendrian embedding $(g_t, G_{s,t})$, using the fact that the space of formal Legendrian embeddings is a Serre fibration over the space of smooth embeddings. Then we may apply the above result to this family of formal Legendrian embeddings. □

7 The plastikstufe

The overtwisted disc described in earlier talks was not the first generalization of the 3-dimensional overtwisted disc to higher dimensions. An earlier generalization (with a definition which is arguably more tractable) was given by Niederkrüger in [Nie06], called the *plastikstufe*. Today we want to define the plastikstufe and point out some features which make it a good candidate for defining overtwistedness in higher dimensions. The plastikstufe was not directly shown to satisfy an *h*-principle, but work of Murphy-Niederkrüger-Plamenevskaya-Stipsicz [MNPS13], Casals-Murphy-Presas [CMP19], and Huang [Hua17] allows us to see that overtwisted contact manifolds in the sense of Borman-Eliashberg-Murphy are precisely those which admit embedded plastikstufe. We give a brief outline of this equivalence at the end of today's talk.

7.1 Definitions

Before defining the plastikstufe, let us define the key properties of the 3-dimensional overtwisted disc which it generalizes.

Definition. A **maximally foliated submanifold** N in a contact manifold (M^{2n+1}, ξ) is a submanifold of dimension $n+1$ on which $\xi|_{TN}$ defines a (possibly singular) foliation.

Definition. An **elliptic singular set** $S \subset N$ in a maximally foliated submanifold of a contact manifold (M, ξ) is a closed, codimension 2 (in N) submanifold which admits a neighborhood diffeomorphic to $\mathbb{D}^2_{x,y} \times S \hookrightarrow N$, and such that $\xi_{TL} = \ker(xdy - ydx)$ on this neighborhood.

Notice that an overtwisted disc D_{ot} in a contact 3-manifold (M, ξ) is maximally foliated by $\xi|_{D_{\text{ot}}}$, and that this singular foliation has a single point for its singular set. Moreover, this singular set is elliptic. These observations motivate the following definition.

Definition. A **plastikstufe** $\mathcal{PS}$ **with core** S in a contact manifold (M^{2n+1}, ξ) is an embedding

$$\iota\colon \mathbb{D}^2 \times S \hookrightarrow M$$

which is maximally foliated by $\iota^*\xi$. The boundary $\partial\mathcal{PS}$ of the plastikstufe is the only closed leaf of the foliation, and $\{0\} \times S$ is an elliptic singular set.

Remark. The plastikstufe is a direct generalization of the overtwisted disc in the sense that $\mathbb{D}_{\text{ot}}$ has an elliptic singular set (a point), and $\mathbb{D}_{\text{ot}}$ consists of a neighborhood of this singular set along with a single closed Legendrian leaf for its boundary. See Figure 9.

Definition. We will call a contact manifold (M^{2n+1}, ξ) **PS-overtwisted** if it contains an embedded plastikstufe of dimension $n+1$.

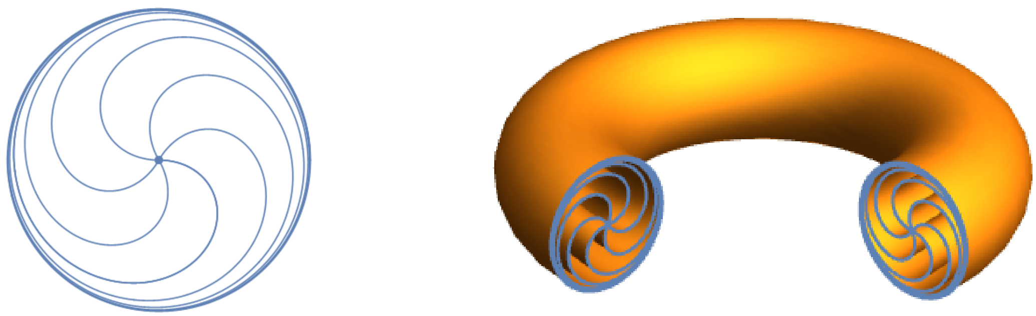
7.2 Obstructing fillability

In addition to the fact that it is a direct generalization of the overtwisted disc in dimension 3, a particularly compelling reason to view the plastikstufe as a good candidate for defining overtwistedness is that PS-overtwisted contact manifolds are not (semipositively) symplectically fillable.

Definition. A symplectic manifold (W, ω) of dimension $2n$ is called **semipositive** if every $A \in \pi_2(M)$ with $\omega(A) > 0$ and $c_1(A) \geq 3-n$ has nonnegative Chern number.

Theorem 7.1 ([Nie06, Theorem 1])**.** *Let (M, ξ) be a closed PS-overtwisted contact manifold. Then M does not have any semipositive symplectic filling.*

Speaker: Randy Van Why



(a) An overtwisted disc. (b) A plastikstufe.

Figure 9: Plastikstufe in dimensions 3 and 5.

Naturally, this is a rather rigid result, in that its proof appeals to the theory of pseudoholomorphic curves and Gromov's compactness theorem. This being a seminar on flexibility, we will only give an impression of the proof. Essentially the same proof which is used to establish non-fillability can be used to show that PS-overtwisted contact manifolds satisfy the Weinstein conjecture.

Theorem 7.2 ([AH09])**.** *Let (M, ξ) be a closed PS-overtwisted contact manifold. Then the Reeb vector field of any contact form inducing ξ has a contractible periodic orbit.*

In both cases, the proof follows the original argument in dimension 3. Let us fix a closed, compact contact manifold $(M^{2n+1}, \xi = \ker\alpha)$ with an embedded plastikstufe $\mathcal{PS} \subset M$, and suppose that (W, ω) is a semipositive symplectic filling of (M, ξ), with compatible almost complex structure J. We will see that no such filling may exist by analyzing a moduli space of J-holomorphic discs in W. In particular, for sets $U \subset W$ and $\tilde{U} \subset \partial W$, we let $\mathcal{M}(U, \tilde{U}, z_0)$ denote the moduli space of J-holomorphic discs lying in U with boundary in $\tilde{U}$, with one marked point $z_0 \in \partial\mathbb{D}^2$. Then we will reach a contradiction by investigating the boundary of $\mathcal{M}(W, \mathcal{PS}, z_0)$.

A first observation is that for any non-constant $u \in \mathcal{M}(W, \mathcal{PS}, z_0)$, $u(\partial\mathbb{D}^2)$ is transverse to the Legendrian foliation on $\mathcal{PS}$. It follows that $u(\partial\mathbb{D}^2)$ is disjoint from $\partial\,\mathcal{PS}$, and also that $u(\partial\mathbb{D}^2)$ is linked with the singular set S. The latter observation allows us to conclude that when we compactify $\mathcal{M}(W, \mathcal{PS}, z_0)$, any constant curve will have its image in S. We have now eliminated two potential boundary components for $\mathcal{M}(W, \mathcal{PS}, z_0)$: there are no non-constant discs touching $\partial\,\mathcal{PS}$, nor any touching the singular set S. A property of compatible almost complex structures on symplectic manifolds with convex boundaries is that J-holomorphic curves cannot be tangent to ∂W. Thus another source of boundary for $\mathcal{M}(W, \mathcal{PS}, z_0)$ is eliminated.

We are now left with two types of boundary elements which could appear in the compactification of $\mathcal{M}(W, \mathcal{PS}, z_0)$: curves which have bubbles at their boundaries, and constant maps. Bubbles can indeed occur — and Niederkrüger addresses them — but let us describe the argument if we assume that no bubbling can happen. Then the compactification of $\mathcal{M}(W, \mathcal{PS}, z_0)$ is a smooth manifold whose boundary elements are constant maps, and these constant maps necessarily have their images in the singular set S. At this point, Niederkrüger uses the assumption that the singular set is elliptic. With this assumption, one can take a small neighborhood U of S in W and a well-chosen almost complex structure J so that the evaluation map ev_{z_0} gives a diffeomorphism between $\mathcal{M}(U, \tilde{U}, z_0)$ and $\tilde{U}$. Here $\tilde{U} = U \cap \mathcal{PS}$. Thus one produces a Bishop family of J-holomorphic discs around the singular set S. Specifically, we have

$$\psi\colon \tilde{U} \to \mathcal{M}(W, \mathcal{PS}, z_0)$$

satisfying $\mathrm{ev}_{z_0} \circ \psi = \mathrm{id}_{\tilde{U}}$, and $\psi(s)$ will be a constant disc, for each $s \in S$.

Finally, we are able to reach our contradiction. For some metric $d(\cdot, \cdot)$ on M and some sufficiently small

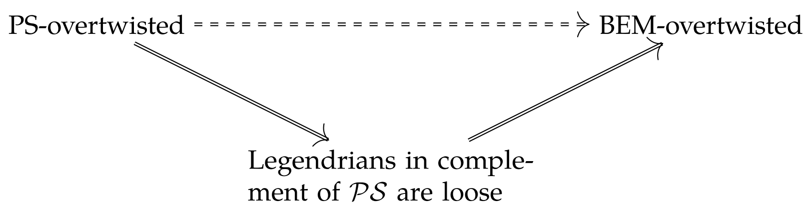
$\epsilon > 0$, we consider the set

$$C_\epsilon := \{\psi(p) | p \in \mathcal{PS}, d(p, S) = \epsilon\}.$$

Under the evaluation map $\mathrm{ev}_{z_0} : \mathcal{M}(W, \mathcal{PS}, z_0) \to \mathcal{PS}$, C_ϵ will map to the generator of $H_n(\mathcal{PS} \setminus S)$, by virtue of the fact that the non-constant discs in our Bishop family link once with the singularity set S. (Recall that $\dim \mathcal{PS} = n + 1$.) But at the same time, C_ϵ is the sole boundary component of the moduli space $\mathcal{M}_\epsilon(W, \mathcal{PS}, z_0) := \mathcal{M}(W, \mathcal{PS}, z_0) - \cup_{r<\epsilon} C_r$. So C_ϵ represents a trivial cycle in the moduli space, contradicting our claim that $[\mathrm{ev}_{z_0}(C_\epsilon)]$ generates the top-degree homology of $\mathcal{PS} - S$. Under the assumption that bubbles cannot occur, this proves Theorem 7.1.

7.3 Plastikstufe and loose Legendrians

In this section, we would like to show that any PS-overtwisted contact manifold is also overtwisted in the sense of [BEM15]. As far as we know, the only currently known proof of this fact relies on first showing that any Legendrian embedding whose complement contains a plastikstufe is loose. Using this fact, Casals-Murphy-Presas establish BEM-overtwistedness of a PS-overtwisted contact manifold.



We will focus today on establishing the first arrow. In particular, we will outline a proof of the following result.

Theorem 7.3 ([MNPS13, Theorem 1.1]). *Let (M^{2n+1}, ξ) be any contact manifold containing a small plastikstufe $\mathcal{PS}$ with spherical core and trivial rotation. Then any Legendrian in (M, ξ) which is disjoint from $\mathcal{PS}$ is loose.*

Remark. A plastikstufe $\mathcal{PS}$ is called *small* if there is an embedded open ball in (M, ξ) containing $\mathcal{PS}$. We will say that $\mathcal{PS}$ has *spherical core* if its singular set is a sphere. The notion of *trivial rotation* is a technical condition which we will not precisely define. All three of these conditions were shown by Huang [Hua17] to be unnecessary, but Huang's proof technique differs from that of Murphy-Niederkrüger-Plamenevskaya-Stipsicz, which we outline here.

The first step taken by Murphy-Niederkrüger-Plamenevskaya-Stipsicz towards proving Theorem 7.3 is to prove the following folklore result, which should be seen as the low-dimensional version of Theorem 7.3.

Theorem 7.4. *Let (M^3, ξ) be an overtwisted contact manifold, and let $L \subset (M, \xi)$ be a Legendrian knot in the complement of an overtwisted disc $\mathbb{D}_{\mathrm{ot}}$. Then a destabilization of L is given by the Legendrian connected sum $L \# \partial \mathbb{D}_{\mathrm{ot}}$.*

Remark. While the proof of Theorem 7.4 is not difficult, it uses a bit of background in convex surface theory which we have not assumed for this seminar.

Proving Theorem 7.3 then basically amounts to carrying out Theorem 7.4 parametrically. In particular, any embedded plastikstufe $\mathcal{PS} \subset (M, \xi)$ with singular set S admits a neighborhood $U_{\mathcal{PS}} \subset (M, \xi)$ which is contactomorphic to

$$(\mathbb{R}^3_{\mathrm{ot}} \times T^*S, \ker(\alpha_{\mathcal{PS}} := \alpha_{\mathrm{ot}} + \lambda_{\mathrm{can}})),$$

where α_{ot} is the standard overtwisted contact form on $\mathbb{R}^3$ and λ_{can} is the canonical 1-form on T^*S. Now for each $s \in S$ we have an overtwisted $\mathbb{R}^3$-slice $\mathbb{R}^3_{\mathrm{ot}} \times \{s\}$, and the slice $\Lambda \cap (\mathbb{R}^3_{\mathrm{ot}} \times \{s\})$ should be a 1-dimensional arc. We then use Theorem 7.4 to identify a zig-zag in the front projection of this arc. If we are able to do this parametrically in s, we should find a region of Λ which looks like a zig-zag crossed with S — that is, Λ will have a loose chart.

Now let $\Lambda_0 \subset \Lambda$ be the region which we will identify as a loose chart. This region must be diffeomorphic to $[0,1] \times S$, and since a loose chart requires a product structure, we must also be able to write $U_{\mathcal{PS}} = \mathbb{R}^3_{\text{ot}} \times T^*S$, with

$$\mathcal{PS} = \mathbb{D}_{\text{ot}} \times \{0\text{-section}\} \quad \text{and} \quad \Lambda_0 = K \times \{0\text{-section}\},$$

where $K \subset \mathbb{R}^3_{\text{ot}}$ is a Legendrian arc. Murphy-Niederkrüger-Plamenevskaya-Stipsicz are able to identify such a neighborhood under the conditions that S is spherical, and that $\mathcal{PS}$ has *trivial rotation*.

Lemma 7.5 ([MNPS13, Lemma 4.7])**.** *Suppose (M^{2n+1}, ξ) is PS-overtwisted and $\Lambda \subset M$ is a given Legendrian. Assume that there is a small plastikstufe $\mathcal{PS} \subset (M, \xi)$ with spherical core and trivial rotation. Then there exists an ambient contact isotopy of M that takes a submanifold Λ_0 of Λ diffeomorphic to $S^{n-1} \times [0,1]$ to a product strip $\Lambda_1 = K \times \{0$ section$\}$ near $\mathcal{PS}$.*

This lemma allows us to apply Theorem 7.4 parametrically. In particular, Lemma 7.5 says that we may take the Legendrian Λ to be given by $K \times S^{n-1}$ in the standard neighborhood $\mathbb{R}^3_{\text{ot}} \times D^*S^{n-1}$ of the plastikstufe. Theorem 7.4 is then applied for each point in S^{n-1} to produce an open subset $V \subset M$ such that $(V, V \cap \Lambda)$ is contactomorphic to $(\mathbb{R}^3_{\text{std}} \times D^*S^{n-1}, \Lambda_0)$, where Λ_0 is the product of a zig-zag in $\mathbb{R}^3_{\text{std}}$ and the zero section of D^*S^{n-1}. Namely, V is a loose chart for Λ, meaning that Λ is loose.

8 Flexible Weinstein structures and overtwistedness

We have recently seen *h*-principles for overtwistedness and loose Legendrian embeddings. In order to connect these two notions, we will pass through *Weinstein cobordisms*, which give us a means of studying symplectic structures via Morse theory. In today's talk we will briefly recall some Weinstein geometry and define *flexible* Weinstein cobordisms. Using these, we prove a geometric criterion for overtwistedness which is stated in terms of loose Legendrian knots.

8.1 Definitions

In this section we recall the notions of Liouville and Weinstein cobordisms, and define what it means for a Weinstein cobordism to be flexible.

Definition. A **Liouville domain** is a triple (W, λ, X_λ), where W is a manifold-with-boundary, $d\lambda$ is a symplectic form on W, and X_λ is a vector field on W which points transversely out of ∂W and satisfies $\iota_{X_\lambda} d\lambda = \lambda$. We call this last condition the **Liouville condition**, and say that X_λ is a **Liouville vector field.** A **Liouville cobordism** is an exact symplectic manifold-with-boundary (W, λ) admitting a vector field X_λ which satisfies the Liouville condition and is transverse to the boundary. We denote by $\partial_+ W$ the **positive boundary** of (W, λ), along which X_λ is outwardly-transverse, and by $\partial_- W$ the **negative boundary**, along which X_λ is inwardly-transverse.

Remark. Notice that a Liouville domain is simply a Liouville cobordism with empty negative boundary. For both Liouville domains and cobordisms, the requirement that X_λ is transverse to ∂W ensures that $\lambda|_{\partial W}$ is a contact form.

Example 8.1. Consider the unit disc $D^{2n} \subset (\mathbb{R}^{2n}, \omega_{\text{std}})$. The 1-form

$$\lambda_{\text{std}} = \frac{1}{2}\Sigma(x_i dy_i - y_i dx_i)$$

is a primitive for the standard symplectic form ω_{std}, and it is straightforward to check that the radial vector field

$$X_{\lambda_{\text{std}}} = \frac{1}{2}\Sigma(x_i \partial_{x_i} + y_i \partial_{y_i})$$

satisfies the Liouville condition. So $(D^{2n}, \lambda_{\text{std}}, X_{\lambda_{\text{std}}})$ is a Liouville domain. By removing a smaller, concentric disc from D^{2n}, we obtain a Liouville cobordism.

Example 8.2. Recall that a hypersurface $\Sigma \subset (M, \xi)$ in a contact manifold is *convex* if there is a vector field v, defined on some neighborhood of Σ, such that $\mathcal{L}_v \xi = \xi$ and v is transverse to Σ. Letting $R_+(\Sigma) \subset \Sigma$ denote the portion of Σ where v is positively transverse to ξ, one can show that $(R_+(\Sigma), \alpha|_\Sigma)$ is a Liouville domain, where $\xi = \ker \alpha$ on M.

We would like to have some sort of handlebody decomposition for symplectic manifolds which is in some sense compatible with Liouville structures. For this reason, we define Weinstein cobordisms to be those Liouville cobordisms which admit a compatible Morse function.

Definition. We call a Liouville cobordism a **Weinstein cobordism** if the Liouville vector field X_λ is gradient-like for some Morse function ϕ.

Example 8.3. Consider the Liouville domain $(D^{2n}, \lambda_{\text{std}}, X_{\lambda_{\text{std}}})$ defined above. The Liouville vector field is gradient-like for the Morse function

$$\phi(x, y) = \frac{1}{4}\sum_{i=1}^{n} x_i^2 + y_i^2$$

(indeed, $X_{\lambda_{\text{std}}}$ is precisely the gradient of ϕ if we use the usual Riemannian structure on D^{2n}), so $(D^{2n}, \lambda_{\text{std}}, X_{\lambda_{\text{std}}})$ is a Weinstein cobordism.

Speaker: Joseph Breen

By requiring that the Liouville vector field be gradient like for a Morse function, we restrict the topology of the underlying manifold. We have the following proposition.

Proposition 8.4. *Let (W, λ, X_λ) be a Weinstein cobordism of dimension $2n$, and let ϕ be a Morse function for which X_λ is gradient-like. Then the critical points of ϕ have index at most n.*

Proof. The key idea here is that the stable manifold of a critical point is isotropic. Let p be a critical point of ϕ, and let q be some point in its stable manifold. We claim that $d\lambda_q = 0$. Indeed, let $\varphi_t : W \to W$ denote the time-t flow of X_λ. The Liouville condition tells us that $\mathcal{L}_{X_\lambda} d\lambda = d\lambda$, and thus $\varphi_t^* d\lambda = e^t d\lambda$, for any t. In particular, we may choose $t \geq 0$ so that $\varphi_t(q) = p$, since q is in the stable manifold of p. Then $\varphi_t^* d\lambda_p = e^t d\lambda_q$, and thus $d\lambda_q = e^{-t}\varphi_t^* d\lambda_p$. By letting t grow without bound, we see that $d\lambda_q = 0$. So the stable manifold of p is isotropic, and thus has dimension at most n. ☐

The upshot is that a Weinstein cobordism admits a handle decomposition with handles of index at most n. Handles of low index will generally create fewer problems for us, so we give them a special name.

Definition. Let (W, λ, X_λ) be a Weinstein cobordism of dimension $2n$, and ϕ a Morse function for which X_λ is gradient-like. We call a critical point p of ϕ **subcritical** if $\mathrm{ind}(p) < n$.

Now the condition that X_λ be gradient-like for ϕ ensures that X_λ is transverse to the regular level sets of ϕ, and thus these submanifolds of W are contact manifolds. The attaching spheres of the handles provided to us by ϕ live in these contact manifolds, and our proof of Proposition 8.4 allows us to see that these attaching spheres are isotropic. In particular, if a critical point is of index n, then the associated attaching sphere will be an isotropic submanifold of dimension $n-1$ in a contact manifold of dimension $2n-1$ — that is a Legendrian. Because we want flexible Weinstein cobordisms to satisfy an h-principle, we are particularly interested in attaching spheres which obey an h-principle. This motivates the following definition.

Definition. Let (W, λ, X_λ) be a Weinstein cobordism of dimension $2n$, and ϕ a Morse function for which X_λ is gradient-like. We call a critical point p of φ **flexible** if the associated attaching sphere is a loose Legendrian, and we say that (W, λ, X_λ) is **flexible** if every critical point of ϕ is either subcritical or flexible.

8.2 Flexible cobordisms preserve overtwistedness

One immediate consequence of the definition of flexible Weinstein cobordisms is that they preserve overtwistedness, in the sense of the following proposition.

Proposition 8.5. *Suppose that (W, λ, X_λ) is a flexible Weinstein cobordism, and that $(\partial_- W, \ker \lambda|_{\partial_- W})$ is an overtwisted contact manifold. Then $(\partial_+ W, \ker \lambda|_{\partial_+ W})$ is overtwisted as well.*

Proof Sketch. Let $\phi : W \to \mathbb{R}$ be a Morse function for which X_λ is gradient-like, with $\partial_- W$ and $\partial_+ W$ being regular level sets of ϕ. Notice that flowing along X_λ gives a contactomorphism between regular level sets of ϕ, provided we do not pass through critical points. So we investigate the effect of passing through critical points, and show that this does not sacrifice overtwistedness. Say we have a critical point contained in the level set $\phi^{-1}(c)$, for some $c \in \mathbb{R}$. Then moving from $\phi^{-1}(c-\epsilon)$ to $\phi^{-1}(c+\epsilon)$ corresponds to performing contact surgery along the isotropic attaching sphere $\Lambda \subset \phi^{-1}(c-\epsilon)$. If $\phi^{-1}(c-\epsilon)$ contains an overtwisted disc, then we can smoothly isotope Λ away from this disc. Because (W, λ, X_λ) is flexible, the attaching spheres of critical points obey an h-principle which allows us to produce a contact isotopy closely approximating this smooth isotopy. That is, we may assume that Λ is disjoint from the overtwisted disc, and when we perform surgery along Λ, the overtwisted disc is unaffected. So if $\phi^{-1}(c-\epsilon)$ is overtwisted, then so is $\phi^{-1}(c+\epsilon)$. Because $\partial_- W$ is overtwisted, we may apply this reasoning at each critical point of X_λ to deduce that $\partial_+ W$ is also overtwisted. ☐

Results such as this one are typical of flexible Weinstein cobordisms, and illustrate the reason they are called flexible: by insisting that the attaching spheres of our handles are loose Legendrians, we buy ourselves a great deal of freedom to isotope these attaching spheres in their respective regular level sets.

8.3 Detecting overtwistedness with loose Legendrians

Finally, we state the main theorem of this talk, which connects looseness for Legendrian embeddings to overtwistedness for contact manifolds, and ties up some loose ends left by the previous two talks.

Theorem 8.6. *Let (M, ξ) be a contact manifold containing a loose Legendrian unknot. Then (M, ξ) is overtwisted.*

One element in the proof of this theorem will be a notion of connected sum for Weinstein cobordisms, which we now define. Let $(W_1, \lambda_1, X_{\lambda_1})$ and $(W_2, \lambda_2, X_{\lambda_2})$ be Weinstein cobordisms with non-empty negative boundary, and let $p_i \in \partial_- W_i$ be a point which is not in the stable manifold of any critical point of X_{λ_i}, for $i = 1, 2$. Then the flowline γ_i of X_{λ_i} which passes through p_i intersects every level set of ϕ_i precisely once, where ϕ_i is a Morse function for which X_{λ_i} is gradient-like. Define a smooth manifold

$$W_1 \overline{\#} W_2 := (W_1 \setminus \mathcal{O}\mathrm{p}(\gamma_1)) \cup (W_2 \setminus \mathcal{O}\mathrm{p}(\gamma_2)),$$

where the gluing is performed to ensure that the Liouville forms and Morse functions map on a collar neighborhood of $\partial\,\mathcal{O}\mathrm{p}(\gamma_1) \cup \partial\,\mathcal{O}\mathrm{p}(\gamma_2)$. Then we have a Weinstein cobordism $(W_1 \overline{\#} W_2, \lambda_1 \overline{\#} \lambda_2, X_{\lambda_1} \overline{\#} X_{\lambda_2})$, with a Morse function $\phi_1 \overline{\#} \phi_2$ regular level sets are the contact connected sums of the regular level sets of ϕ_1 and ϕ_2. We call this Weinstein cobordism the **vertical connected sum** of $(W_1, \lambda_1, X_{\lambda_1})$ and $(W_2, \lambda_2, X_{\lambda_2})$.

The proof will also make use of a type of cylindrical Weinstein cobordism over contact manifolds.

Definition. Let (M, ξ) be a contact manifold, with $\xi = \ker \alpha$ on M. Then $\lambda := e^s \alpha$ gives a Liouville form on $W := [0, 1]_s \times M$, and we call (W, λ, X_λ) the (**compact**) **symplectization** of (M, ξ).

Remark. The Liouville vector field on the symplectization is ∂_s, which is clearly gradient-like for the function $\phi = s$, and so the symplectization is a Weinstein cobordism.

Proposition 8.7. *Let (M, ξ) be a contact manifold containing a loose Legendrian unknot. Then for any Weinstein cobordism W, the vertical connected sum $W \overline{\#} S(M)$ is flexible.*

Proof Sketch. First, note that the critical set of $W \overline{\#} S(M)$ is precisely that of W, because the compact symplectization of a contact manifold has no critical points. So we need to show that every critical point of W of index n (where $\dim W = 2n$) is flexible in $W \overline{\#} S(M)$, though perhaps not so in W. To this end, let Λ be the attaching sphere associated to p, and let $\Lambda_0 \subset (M, \xi)$ be a loose Legendrian. Now we may choose a loose chart U for Λ_0 which avoids Λ, since $\Lambda_0 \subset (M, \xi)$ and p comes from W. This is possible because the gluing region for the vertical connected sum $W \overline{\#} S(M)$ avoids the stable manifolds of the critical points of p. Notice that U witnesses the looseness of the Legendrian $\Lambda \# \Lambda_0$, and that since Λ_0 is an unknot, $\Lambda \# \Lambda_0$ is smoothly isotopic to Λ. Moreover, because Λ is loose, we may apply the *h*-principle for loose Legendrians to make this a contact isotopy. So Λ is loose in $W \overline{\#} S(M)$, and we conclude that $W \overline{\#} S(M)$ is flexible. ☐

Proof Idea for Theorem 8.6. The proof relies on the following claim which we will not justify today:

> **Claim.** There is a Weinstein cobordism (W, λ, X_λ) with negative boundary $(S^{2n-1}, \xi_{\mathrm{ot}})$ and positive boundary $(S^{2n-1}, \xi_{\mathrm{std}})$.

This cobordism can be constructed explicitly, using open book decompositions for the contact structures ξ_{ot} and ξ_{std} on S^{2n-1}. According to the previous proposition, the vertical connected sum $W \overline{\#} S(M)$ is flexible. In particular, we have a flexible Weinstein cobordism from

$$\partial_-(W \overline{\#} S(M)) = \partial_- W \# M = (S^{2n-1}, \xi_{\mathrm{ot}}) \# M,$$

which is overtwisted, to

$$\partial_+(W \overline{\#} S(M)) = \partial_+ W \# M = (S^{2n-1}, \xi_{\mathrm{std}}) \# M \cong M.$$

By Proposition 8.5, (M, ξ) is overtwisted. ☐

10 Overtwistedness of $(+1)$-surgeries

The goal of today's talk is to present a surgery criterion for overtwistedness of a contact manifold. In dimension three, contact (± 1)-surgeries are relatively well-understood; contact (-1)-surgery — also known as *Legendrian surgery* — is known to preserve tightness, while the following proposition shows that performing contact $(+1)$-surgery on a tight contact manifold can have an overtwisted outcome.

Proposition 10.1. *Let $L \subset (M', \xi')$ be a Legendrian knot in a contact 3-manifold (M', ξ'), and let (M, ξ) be the contact manifold which results from contact $(+1)$-surgery along $S_{\pm}(L)$. Then (M, ξ) is overtwisted.*

We will provide a proof sketch for Proposition 10.1 after reviewing the definition of contact $(+1)$-surgery.

Recall that a Legendrian knot is *stabilized* by implanting a zig-zag (as in Figure 6) in the front projection of the knot. It is natural to think of loose Legendrians in higher dimensions as those Legendrians which have been stabilized. The stabilization procedure introduces a loose chart into a Legendrian submanifold, creating a loose Legendrian embedding, and the loose Legendrians are precisely those Legendrians which can be presented in this manner. A natural generalization (and strengthening) of Proposition 10.1 to higher dimensions is thus the following.

Theorem 10.2 ([CMP19, Theorem 1.1])**.** *Let (M, ξ) be a contact manifold of dimension at least 5. Then (M, ξ) is overtwisted if and only if there exists a contact manifold (M', ξ') and a loose Legendrian sphere $\Lambda \subseteq (M', \xi')$ such that (M, ξ) is contactomorphic to the contact $(+1)$-surgery of (M', ξ') along Λ.*

Concluding overtwistedness from the criterion stated in Theorem 10.2 is the more difficult direction of the proof, so we address this first (after some background material on contact surgery). Once we have established this direction, we recall the notion of compatibility between open book decompositions and contact structures before showing that all overtwisted contact manifolds in dimension at least five can be presented as $(+1)$-surgeries as described above[6].

10.1 Background

Before proving Theorem 10.2, we give some background on performing contact $(+1)$-surgery in arbitrary dimension. A good resource for this material is [Avd12, Section 9].

10.1.1 Generalized Dehn twists

The generalized Dehn twist will be a symplectomorphism $\tau_n \colon T^*S^n \to T^*S^n$, generalizing the familiar Dehn twist on the annulus. We identify T^*S^n as

$$T^*S^n = \{(u, v) \in \mathbb{R}^{2(n+1)} | \|u\| = 1, \langle u, v \rangle = 0\},$$

and the canonical Liouville form $\lambda_{\text{std}} = \frac{1}{2}\Sigma_{i=1}^{n+1}(u_i dv_i - v_i du_i)$ on $\mathbb{R}^{2(n+1)}$ restricts to a Liouville form on T^*S^n. Now for some small $\epsilon \ll 1$, define a smooth function $f \colon [0, \infty) \to \mathbb{R}$ satisfying

1. $f(0) = \pi$, and $f^{(k)}|_{(-\epsilon,\epsilon)} \equiv 0$, for all $k \geq 1$;
2. f is non-decreasing;
3. $f|_{[2\epsilon,\infty)} \equiv 2\pi$.

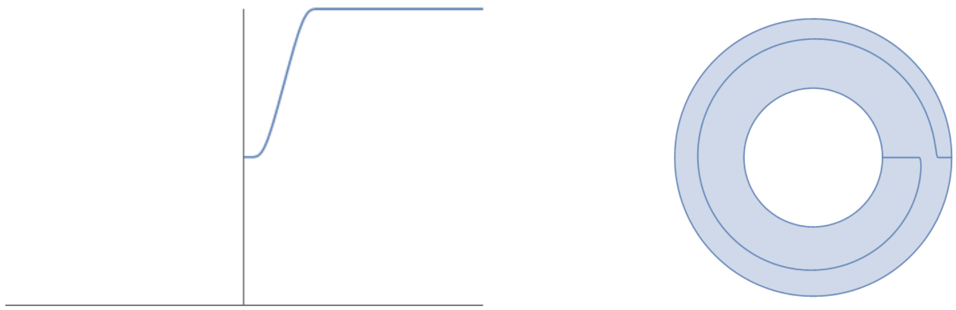
Then the diffeomorphism $\tau_n \colon T^*S^n \to T^*S^n$ is defined by

$$\tau_n(u, v) = (\cos(f(\|v\|)) \cdot u + \sin(f(\|v\|)) \cdot \frac{v}{\|v\|}, -\|v\| \sin(f(\|v\|)) \cdot u + \cos(f(\|v\|)) \cdot v).$$

Notice that $\tau_1 \colon T^*S^1 \to T^*S^1$ gives the usual Dehn twist.

Speaker: Austin Christian

[6] Caveat: We will show that overtwisted contact manifolds are presented as contact $(+1)$-surgeries, but we will not verify that the Legendrian along which surgery is performed is loose.



(a) $f\colon [0,\infty) \to \mathbb{R}$ (b) A Dehn twist

Figure 10: The function f and a Dehn twist it produces.

Theorem 10.3 ([Sei99, Lemma 6.2])**.** *The diffeomorphism τ_n preserves the symplectic form $d\lambda_{\mathrm{std}}$, and the isotopy class of τ_n in $\mathrm{Symp}(D^*S^n, \partial D^*S^n)$ is independent of ϵ and f.*

Remark. While τ_n preserves $d\lambda_{\mathrm{std}}$, the Liouville form λ_n is not preserved. This creates a subtlety in the definition of contact $(+1)$-surgery which we will ignore.

Definition. Any symplectomorphism which is symplectically isotopic (rel. boundary) to τ_n is called a **generalized Dehn twist** of T^*S^n.

10.1.2 Contact $(+1)$-surgery

We can now describe contact $(+1)$-surgery on a Legendrian sphere. Any Legendrian sphere $\Lambda^n \subset (M^{2n+1}, \xi)$ admits a neighborhood $N(\Lambda)$ which is contactomorphic to the 1-jet space of S^n:

$$N(\Lambda) \cong (J^1S^n, \ker\alpha_{\mathrm{std}}) = (\mathbb{R}_z \times T^*S^n, \ker(dz - \lambda_{\mathrm{std}})).$$

We obtain a new contact manifold (M', ξ') by first removing the portion of $N(\Lambda)$ identified with $(0,1)\times D^*S^n$, and then gluing according to

$$(x,0) \sim (\tau_n^{-1}(x), 1) \text{ for } x \in D^*S^n \quad \text{and} \quad (x,t) \sim (x,t') \text{ for } x \in \partial D^*S^n.$$

Because the support of τ_n is some very small subset of D^*S^n, this gluing produces a smooth manifold. Moreover, the fact that τ_n is a symplectomorphism will ensure that (M', ξ') is contact.

Definition. We call the contact manifold (M', ξ') defined above the **contact** $(+1)$**-surgery of** (M, ξ) **along** Λ.

In general, the contactomorphism type of the contact manifold resulting from this construction depends on the Legendrian isotopy type of the parametrization $S^n \to \Lambda$. However, we will only perform surgery on loose Legendrian spheres today, and any two parametrizations of a loose Legendrian sphere are Legendrian isotopic.

10.2 Contact $(+1)$-surgery is overtwisted

In this section we prove one half of Theorem 10.2, in the form of the following proposition.

Proposition 10.4. *Let (M', ξ') be a contact manifold of dimension at least 5, and let (M, ξ) be the contact manifold which results from contact $(+1)$-surgery of (M', ξ') along a loose Legendrian sphere $\Lambda \subseteq (M', \xi')$. Then (M, ξ) is overtwisted.*

A glib summary of the proof of Proposition 10.4 found in [CMP19] is as follows: each point in the equator of the Legendrian sphere Λ on which we perform contact $(+1)$-surgery has an associated overtwisted disc in (M, ξ), and a parametrized family of overtwisted discs is a plastikstufe.

To make this summary somewhat rigorous, let's first prove Proposition 10.1.

Proof sketch for Proposition 10.1. This proof can be found in [DGS04, Section 1]. Given $L \subset (M', \xi')$, let $L' \subset (M', \xi')$ be a transverse pushoff of L. The linking number of L and L' is, by definition, the Thurston-Bennequin number $\mathrm{tb}(L)$ of L. Now consider performing a contact $(+1)$-surgery along $S_\pm(L)$. Smoothly, this is a $(\mathrm{tb}(S_\pm(L))+1)$-surgery, which is to say a $\mathrm{tb}(L)$-surgery, since $\mathrm{tb}(S_\pm(L)) = \mathrm{tb}(L) - 1$. Now $S_\pm(L)$ and L' cobound a Seifert surface Σ, and the framing of $S_\pm(L)$ determined by Σ is $\ell k(S_\pm(L), L') = \ell k(L, L') = \mathrm{tb}(L)$. Because this agrees with the framing coefficient of the surgery, the meridional disc arising from the surgery will join with Σ to produce an embedded disc D in (M, ξ) whose boundary is L'. The contact framing of L' is $\mathrm{tb}(L)$ and the surface framing determined by D is also $\mathrm{tb}(L)$. We conclude that D is an overtwisted disc. ☐

Next we turn to the proof of Proposition 10.4. First, we choose a Legendrian sphere $\tilde{\Lambda}$ whose spherical stabilization is Λ — this can be done because Λ is loose. By identifying a neighborhood of $\tilde{\Lambda}$ with $(J^1S^n, \ker \alpha_{\mathrm{std}})$, we may realize Λ as the spherical stabilization of the zero section in J^1S^n over its equator $S^{n-1} \subset S^n$. Now for each point x in the equator, we define the circle $S^1_x \subset S^n$ to be the unique geodesic (i.e., meridian) passing through x and the north and south poles. Then we have a contact submanifold

$$(J^1S^1_x, \ker \alpha_{\mathrm{std}}) \subset (J^1S^n, \ker \alpha_{\mathrm{std}})$$

of dimension three, and $\Lambda \cap J^1S^1_x$ is the stabilization of the zero section of $J^1S^1_x$.

Our next observation is that the symplectic submanifold $T^*S^1_x \subset T^*S^n$ is preserved by the generalized Dehn twist $\tau_n \colon T^*S^n \to T^*S^n$, because $S^1_x \subset S^n$ is a geodesic, and the Dehn twist is defined using geodesic flow. It follows that the image of $J^1S^1_x \subset (M', \xi')$ in (M, ξ) is contactomorphic to $(+1)$-contact surgery of $J^1S^1_x$ along $\Lambda \cap J^1S^1_x$; we call this image (M_x, ξ). But since $\Lambda \cap J^1S^1_x$ is a stabilized Legendrian, Proposition 10.1 tells us that (M_x, ξ) is overtwisted, for every $x \in S^{n-1}$. The family of overtwisted discs produced by the proof of Proposition 10.1 constitutes a plastikstufe with spherical core. Applying Theorem 7.3, we see that (M, ξ) is overtwisted.

Remark. Technically we should also show that this platikstufe is small and has trivial rotation class, but these are notions we side-stepped before, so we will continue to do so.

10.3 Supporting open book decompositions

We will show that overtwisted contact manifolds may be presented as contact $(+1)$-surgeries by investigating their supporting open book decompositions. First, we quickly recall some open book decomposition notions.

We suppose that (W, λ) is a Liouville domain, and that $\varphi \colon W \to W$ is an exact symplectomorphism supported away from ∂W. In particular, $\varphi^*\lambda = \lambda + dh$ for some smooth function $h \colon W \to \mathbb{R}$ supported away from ∂W. The Liouville domain (W, λ) will serve as the *page* of our open book decomposition, and φ will be the *monodromy*. We let

$$M := ((W \times [0,1])/(x,1) \sim (\varphi(x), 0)) \cup_{\partial W \times S^1} \left(\partial W \times D^2\right).$$

By choosing $K \in \mathbb{R}$ sufficiently large, we can ensure that

$$\xi := \ker((\lambda + Kd\theta + \theta dh) \cup (\lambda|_{\partial W} + Kr^2 d\theta))$$

is a contact structure on M, and we define $\mathrm{OB}(W, \lambda, \varphi) := (M, \xi)$. We say that the open book decomposition (W, λ, φ) *supports* the contact structure ξ.

A celebrated result of Giroux says that every contact manifold admits a supporting open book.

Theorem 10.5 ([Gir02, Theorem 10])**.** *Every contact manifold (M, ξ) is contactomorphic to some $\mathrm{OB}(W, \lambda, \varphi)$, with (W, λ) a Weinstein domain.*

Notice that if $\psi \colon W \to W$ is a symplectomorphism, then $\mathrm{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1}) \cong \mathrm{OB}(W, \lambda, \varphi)$. A particular monodromy we would like to consider is the case where $\varphi = \tau_L$ is a Dehn twist about a Lagrangian sphere $L \subset (W, \lambda)$. For $\dim(L)$ at least two, a Lagrangian sphere is necessarily an exact Lagrangian, meaning that

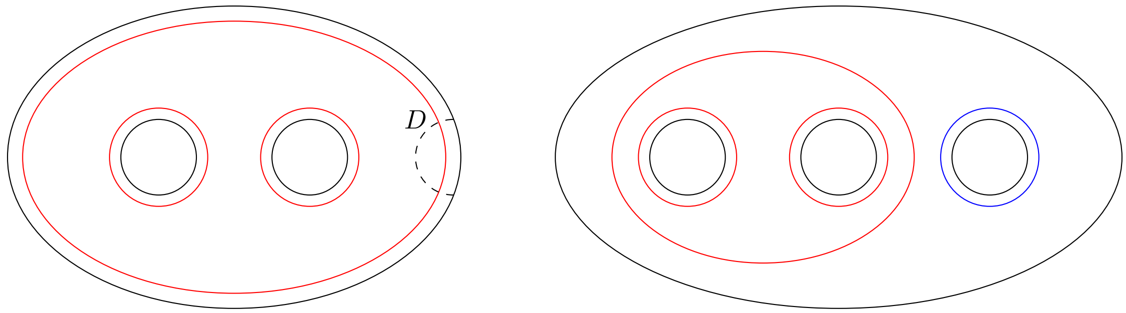


Figure 11: An open book decomposition for $(L(3,1), \xi_{\text{std}})$, and a stabilization of this OBD. Each red circle represents a right-handed Dehn twist. The stabilization is positive or negative depending on whether we take the Dehn twist about the blue circle to be right- or left-handed.

$\lambda|_L = df$ for some $f\colon W \to \mathbb{R}$. Up to Liouville homotopy, we may rescale $\lambda|_L$ so that $f\colon W \to [-\epsilon, \epsilon]$, and we then use f to lift L to a Legendrian sphere $\Lambda \subset \mathrm{OB}(W, \lambda, \varphi)$. We write

$$(M, \xi, \Lambda) = \mathrm{OB}(W, \lambda, \varphi, L)$$

to indicate that (W, λ, φ) supports (M, ξ), and that $\Lambda \subset (M, \xi)$ is a Legendrian sphere which is Legendrian isotopic to the Legendrian lift of $L \subset (W, \lambda)$, an exact Lagrangian sphere.

The following is a well-known relationship between Dehn twists along exact Lagrangian spheres in the page of an open book decomposition and contact surgeries on the corresponding contact manifold.

Proposition 10.6. *Suppose that $(M, \xi, \Lambda) = \mathrm{OB}(W, \lambda, \varphi, L)$. Then the contact manifold $\mathrm{OB}(W, \lambda, \varphi \circ \tau_L)$ is obtained from (M, ξ) by contact (-1)-surgery along Λ, and $\mathrm{OB}(W, \lambda, \varphi \circ \tau_L^{-1})$ is obtained via contact $(+1)$-surgery along Λ.*

We will show that overtwisted contact manifolds are $(+1)$-surgeries by identifying a negative Dehn twist in a supporting open book decomposition. This negative Dehn twist is found in a *negative stabilization* of a given supporting open book decomposition for the contact manifold, so we now describe the stabilization process.

An open book (W, λ, φ) is stabilized along a Lagrangian disc $D \subset (W, \lambda)$ with Legendrian boundary $\partial D \subset (\partial W, \ker \lambda)$ by attaching a Weinstein n-handle to (W, λ) along ∂D. The page of the new open book decomposition is $(W \cup H, \lambda')$. This page contains a Lagrangian sphere L whose lower hemisphere is D and whose upper hemisphere is the core of the handle H; we call the open book decomposition $(W \cup H, \lambda', \varphi \circ \tau_L)$ the *positive stabilization* of (W, λ, φ), and call $(W \cup H, \lambda', \varphi \circ \tau_L^{-1})$ the *negative stabilization* of (W, λ, φ) along D.

Finally, let us define a particular contact structure ξ_- on S^{2n+1} via an open book. Letting $\tau_n\colon T^*S^n \to T^*S^n$ be the generalized Dehn twist described above, we define

$$(S^{2n+1}, \xi_-) := \mathrm{OB}(T^*S^n, \lambda_{\text{std}}, \tau_n^{-1}).$$

We can now describe the negative stabilization of (M, ξ) via a contact connected sum.

Theorem 10.7 (Giroux)**.** *Let $(M, \xi) = \mathrm{OB}(W, \lambda, \varphi)$ be a contact manifold, and let $D \subset (W, \lambda)$ any Lagrangian disc with Legendrian boundary $\partial D \subset (\partial W, \ker \lambda)$. Then the positive stabilization of $\mathrm{OB}(W, \lambda, \varphi)$ along D is contactomorphic to (M, ξ), and the negative stabilization of $\mathrm{OB}(W, \lambda, \varphi)$ along D is contactomorphic to the contact connected sum $(M \# S^{2n+1}, \xi \# \xi_-)$.*

10.4 Overtwisted manifolds are $(+1)$-surgeries

Finally we show that if (M, ξ) is an overtwisted contact manifold of dimension at least 5, then there exists a contact manifold (M', ξ') and a loose Legendrian sphere $\Lambda \subseteq (M', \xi')$ such that (M, ξ) is contactomorphic to the contact $(+1)$-surgery of (M', ξ') along Λ. We do this by showing that (M, ξ) is supported by a negatively stabilized open book decomposition and then applying Proposition 10.6.

Recall that the existence h-principle for overtwisted contact structures allows us to choose a contact structure $\tilde{\xi}$ on M in any homotopy class of almost contact structures. In particular, we may choose $\tilde{\xi}$ so that $(M \# S^{2n+1}, \tilde{\xi} \# \xi_-)$ is in the same homotopy class as (M, ξ). The uniqueness portion of the h-principle then ensures that $\tilde{\xi} \# \xi_-$ is isotopic to ξ, since these are overtwisted contact structures in the same homotopy class of almost contact structures.

Next, Theorem 10.5 allows us to choose an open book (W, λ, φ) supporting $(M, \tilde{\xi})$, and Theorem 10.7 tells us that the negative stabilization of $\mathrm{OB}(W, \lambda, \varphi)$ is contactomorphic to $(M \# S^{2n+1}, \tilde{\xi} \# \xi_-)$, and hence to (M, ξ). So (M, ξ) is supported by a negatively stabilized open book decomposition. But this means that the monodromy of the open book decomposition includes a negative Dehn twist and thus, by Proposition 10.6, (M, ξ) is the result of some $(+1)$-contact surgery.

Remark. Proposition 10.6 essentially establishes an equivalence between the existence of a negatively stabilized open book decomposition supporting (M, ξ) and our ability to realize (M, ξ) as the result of $(+1)$-surgery along a Legendrian in some other contact manifold. By doing quite a bit more work, one can show that the standard Legendrian unknot in (S^{2n+1}, ξ_-) is loose, and thus the Legendrian along which we do surgery to obtain (M, ξ) is loose. The upshot is that the existence of a negatively stabilized supporting open book gives another characterization of overtwistedness.

10.5 Geometric criteria for overtwistedness

We briefly wrap up the second half of the quarter with the following theorem characterizing overtwistedness.

Theorem 10.8 (c.f. [CMP19, Theorem 1.1])**.** *Let (M, ξ) be a contact manifold of dimension at least 5. Then the following are equivalent.*

1. *The contact manifold (M, ξ) is overtwisted.*
2. *The standard Legendrian unknot in (M, ξ) is a loose Legendrian submanifold.*
3. *There exists a small plastikstufe with spherical core and trivial rotation in (M, ξ).*
4. *There exists a contact manifold (M', ξ') and a loose Legendrian submanifold $\Lambda \subset (M', \xi')$ such that (M, ξ) is contactomorphic to the contact $(+1)$-surgery of (M', ξ') along λ.*
5. *There exists a negatively stabilized contact open book decomposition compatible with (M, ξ).*

Each of these characterizations has their own uses. The very definition of overtwistedness gives us the desired h-principle, but does not provide a tractable means of verifying overtwistedness. The existence of a plastikstufe allows us to prove the Weinstein conjecture for overtwisted contact manifolds and to obstruct fillability, but does not directly give the desired h-principle. Our other three characterizations — in terms of loose Legendrian unknots, surgery along loose Legendrians, and negatively stabilized open books — each give us practical means of verifying the overtwistedness of a given contact manifold.

References

[AH09] Peter Albers and Helmut Hofer. On the Weinstein conjecture in higher dimensions. *Comment. Math. Helv.*, 84(2):429–436, 2009.

[Avd12] Russell Avdek. Liouville hypersurfaces and connect sum cobordisms. *arXiv preprint arXiv:1204.3145*, 2012.

[BEM15] Matthew Strom Borman, Yakov Eliashberg, and Emmy Murphy. Existence and classification of overtwisted contact structures in all dimensions. *Acta Math.*, 215(2):281–361, 2015.

[Ben83] Daniel Bennequin. Entrelacements et equations de Pfaff. *Astérique*, 107:87–161, 1983.

[CMP19] Roger Casals, Emmy Murphy, and Francisco Presas. Geometric criteria for overtwistedness. *J. Amer. Math. Soc.*, 32(2):563–604, 2019.

[CPP19] Roger Casals, Dishant M Pancholi, and Francisco Presas. The Legendrian Whitney Trick. *arXiv preprint arXiv:1908.04828*, 2019.

[DGS04] Fan Ding, Hansjörg Geiges, and András I Stipsicz. Surgery diagrams for contact 3-manifolds. *Turkish J. Math.*, 28(1):41–74, 2004.

[EH01] John B Etnyre and Ko Honda. On the nonexistence of tight contact structures. *Ann. of Math.*, pages 749–766, 2001.

[Eli81] Yakov Eliashberg. Rigidity of symplectic and contact structures. *preprint*, 1981.

[Eli89] Yakov Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989.

[Eli15] Yakov Eliashberg. Recent advances in symplectic flexibility. *Bull. Amer. Math. Soc.*, 52(1):1–26, 2015.

[EM02] Yakov Eliashberg and Nikolai Mishachev. *Introduction to the h-principle*. Number 48. American Mathematical Soc., 2002.

[EM09] Yakov Eliashberg and Nikolai Mishachev. Wrinkled embeddings. *Contemp. Math.*, 498:207–232, 2009.

[Etn10] John B Etnyre. On knots in overtwisted contact structures. *arXiv preprint arXiv:1012.3745*, 2010.

[Gei08] Hansjörg Geiges. *An Introduction to Contact Topology*, volume 109. Cambridge University Press, 2008.

[Gir02] Emmanuel Giroux. Géométrie de contact: de la dimsnion trois vers les dimensions supérieures. In *Proceedings of the ICM*, volume 2, pages 405–414, Beijing 2002.

[Gro69] Mikhail Leonidovich Gromov. Stable mappings of foliations into manifolds. *Mathematics of the USSR-Izvestiya*, 3(4):671, 1969.

[Gro85] Mikhael Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Gro86] Mikhael Gromov. *Partial Differential Relations*. Ergeb. Math. Grenzgeb.(3), 1986.

[HH18] Ko Honda and Yang Huang. Bypass attachments in higher-dimensional contact topology. *arXiv preprint arXiv:1803.09142*, 2018.

[Hon] Ko Honda. Notes for Math 599: Contact geometry. https://www.math.ucla.edu/~honda/math599/notes.pdf.

[Hua17] Yang Huang. On plastikstufe, bordered Legendrian open book and overtwisted contact structures. *J. Topol.*, 10(3):720–743, 2017.

[Lut71] R Lutz. Sur quelques propriétés des formes différentielles en dimension trois. *Thèse, Strasbourg*, 1971.

[Mar71] Jean Martinet. Formes de contact sur les variétés de dimension 3. In *Proceedings of Liverpool Singularities Symposium II*, pages 142–163. Springer, 1971.

[MNPS13] Emmy Murphy, Klaus Niederkrüger, Olga Plamenevskaya, and András I Stipsicz. Loose Legendrians and the plastikstufe. *Geom. Topol.*, 17(3):1791–1814, 2013.

[MNW13] Patrick Massot, Klaus Niederkrüger, and Chris Wendl. Weak and strong fillability of higher dimensional contact manifolds. *Invent. Math.*, 192(2):287–373, 2013.

[MS14] Stefan Müller and Peter Spaeth. Gromov's alternative, Eliashberg's shape invariant, and c^0-rigidity of contact diffeomorphisms. *Internat. J. Math.*, 25(14):1450124, 2014.

[Mur12] Emmy Murphy. Loose Legendrian embeddings in high dimensional contact manifolds. *arXiv preprint arXiv:1201.2245*, 2012.

[Nie06] Klaus Niederkrüger. The plastikstufe–a generalization of the overtwisted disk to higher dimensions. *Algebr. Geom. Topol.*, 6(5):2473–2508, 2006.

[Sei99] Paul Seidel. Lagrangian two-spheres can be symplectically knotted. *J. Differential Geom.*, 52:145–171, 1999.